Solving a Location Problem of a Stackelberg Firm Competing with Cournot-Nash Firms

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Abstract

We study a discrete facility location problem on a network, where the locating firm acts as the leader and other competitors as the followers in a Stackelberg-Cournot-Nash game. To maximize expected profits the locating firm must solve a mixed-integer problem with equilibrium constraints. Finding an optimal solution is hard for large problems, and full-enumeration approaches have been proposed in the literature for similar problem instances. We present a heuristic solution procedure based on simulated annealing. Computational results are reported.

Keywords: Location Analysis; Stackelberg-Cournot-Nash Equilibrium; Game Theory; Variational Inequality; Simulated Annealing

1 Introduction

The choice of locations for manufacturing facilities is an important strategic decision. Such choices involve large expenditures of capital and are difficult and expensive to change. A poor choice will involve significant negative consequences as the manufacturer either suffers the expense of relocating a facility or else lives with ongoing disadvantages brought about by the suboptimal location of that facility. Because of the importance of the problem, it has been

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extensively studied and the associated literature is vast. A good overall survey is Hale and Moberg (2003). A significant portion of the literature is concerned with competitive facility location, *i.e.* locating facilities so as to maximize profits or market share in a competitive situation. Some relevant research articles and surveys of the competitive facility location literature are Friesz et al. (1988), Eiselt et al. (1993), Serra and ReVelle (1999), Plastria (2001), Plastria and Vanhaverbeke (2007), Santos-Peñate et al. (2007), and most recently Kress and Pesch (2011). However much of the literature dealing with facility location subject to competition deals with the location of retail or similar facilities where customers are distributed in some way and will choose a conveniently located facility following some rule *e.g.* a "gravity model". (A review of such literature is found in Dobson and Kamarkar, 1985.) Models appropriate to the location of manufacturing facilities are necessarily different. Rather than placing retail or similar facilities in proximity to demand points, so as to maximize utilization or market share (essentially competing on the basis of proximity), the firm places its manufacturing facilities so as to be able to compete on the basis of cost. That is, facilities are located so as to enable the firm to deliver its products to market as cheaply as possible, by controlling both manufacturing costs, which may vary from candidate location to candidate location, and transportation costs from the manufacturing facility to the markets it serves.

Another factor which must be taken into account is market competition. If the market for a particular commodity is controlled by a small number of competing firms (oligopoly) then a firm must take into account the reactions of its competitors to any planned move. (In a monopolistic situation there is no competition to consider, and where the number of competitors is sufficiently large the actions of competitors do not need to be taken into account.) A firm must expect that its competitors will react to whatever it does (in such a way as to maximize their own profits), and by their actions change market conditions, especially prices, by increasing or decreasing the supply of goods available at different markets. In order to evaluate a potential strategy, a firm must calculate how the competition will react to that strategy. This calculation may be done by computing an equilibrium, termed a "Cournot-Nash-Stackelberg" equilibrium (Sherali et al., 1983). The firm choosing a site for a new facility (referred to as the "Stackelberg player") first makes its choice of sites for establishing manufacturing facilities, what production levels to maintain at each facility and where to sell the materials so produced. Its competitors ("Cournot players") react, maximizing their own profits, until an equilibrium is reached. In order to find the best possible solution to the manufacturing facility location problem, a firm must therefore solve an optimization problem which incorporates a set of equilibrium constraints to model the response by the competition. See Miller et al. (1996) for more discussion on this topic.

The basic structure of the manufacturing facility location problem is therefore to choose locations for manufacturing facilities such that the firm's profits are maximized at the resulting market equilibrium. The literature on the subject of market equilibrium modeling and its computation and analysis is fairly substantial. Among many research papers on the oligopolistic competition models since Cournot (1838), we want to mention two particular papers. Miller et al. (1991) considered a variational inequality formulation of an oligopolistic competition where firms compete with production quantities and and shipping amounts between markets and production facilities. Miller et al. (1991) assumed that the market price is determined by an inverse demand function, as in Cournot (1838). Later, Friesz et al. (2006) extended the model to consider dynamic environments and formulated the equilibrium problem as a differential variational inequality.

On the other hand, when we consider spatially separated markets for a single product, there is another approach to determine the market price of the product. It is based on the spatial price equilibrium models of Samuelson (1952) and Takayama and Judge (1971). Variational inequalities, or some other equivalent forms, are typically used for modeling the oligopolistic equilibrium. Among many approaches that use variational inequalities for analysis and computation, there are Dafermos and Nagurney (1984), Harker (1986), and Nagurney (1987) in this category. The existence and uniqueness results were provided for the variational inequality models, and the computational results are also provided. See Nagurney (1998) for review.

The relevant literature that considers location decisions in the form of Stackelberg-

Cournot-Nash games begins with Sherali et al. (1983), who considers firms that compete on production quantities. They extended the classic Cournot games by introducing a Stackelberg firm who determines the production quantity by explicitly considering the other firms reaction. In their model, the market price was determined by an inverse demand function based on the total market production quantity. The firm who introduces a new facility is regarded as a leader, and all other market oligopolists are followers; therefore, Stackelberg-Cournot-Nash games are often called leader-follower games.

In the line of the location decision model of Sherali et al. (1983) and the spatial model of Miller et al. (1991), Tobin et al. (1995) consider a Stackelberg-Cournot-Nash game where a leader firm introduces a new facility and other firms compete with production quantities and shipping decisions to markets. Later, Miller et al. (2007) further extends to consider a multi-period decision making problem. All three papers modeled the market prices using inverse demand functions.

This category of problems is of this paper's interest. In particular, we will use the modeling framework of Tobin et al. (1995), and propose a heuristic algorithm based on simulated annealing. While the underlying oligopoly models are well studied both analytically and computationally, the location decision problems are generally very hard to solve. Usually, such a problem is modeled as a mathematical program with equilibrium constraints (MPEC) (Luo et al., 1996). Both Tobin et al. (1995) and Miller et al. (2007) used a full enumeration of all possible locations to determine an optimal solution that maximizes the leader firm's profit resulting from the competition. In this paper, we will investigate if an optimal solution could be obtained more efficiently.

There are also location-problem counterparts for the spatial price equilibrium models. Tobin and Friesz (1986) built a location decision making problem based on the spatial price equilibrium of Friesz et al. (1984). Friesz et al. (1989) provided existence results, and Miller et al. (1991) proposed a heuristic method that approximate the spatial price equilibrium processes, which uses sensitivity analysis of the market equilibrium to determine an optimal location. The method requires solving a nonlinear integer programming problem in each

Table 1: A Summary of Relevant Models in the Literature

	Market Price Model	
Competition Model	Inverse Demand	Spatial Price Equilibrium
Cournot-Nash (quantity)	Cournot (1838), Harker (1984)	N/A
Cournot-Nash (quantity, shipping)	Harker (1986), Miller et al. (1991), Friesz et al. (2006)†	Dafermos and Nagurney (1984), Friesz et al. (1984), Harker (1986), Nagurney (1987)
Stackelberg-Cournot-Nash (quantity)	Sherali et al. (1983)	N/A
Stackelberg-Cournot-Nash (quantity, shipping, location)	Tobin et al. (1995), Miller et al. (2007)†, *this paper*	Tobin and Friesz (1986), Friesz et al. (1989), Miller et al. (1992)

† dynamic models

main iteration.

A summary of the above mentioned relevant models is provided in Table 1. As we have already discussed above, we consider two factors. The first column in the table represents research that uses inverse demand functions to model market prices depending on the total supply to the markets, while the second column represents research that uses spatial price equilibrium models using the Takayama and Judge (1971) approach. The first row is the Cournot-Nash model by Cournot (1838), and the second row provides Cournot-Nash models with spatially separated markets and shipping decisions. The third row is the Stakelberg-Cournot-Nash extension of Cournot (1838), and the last row provides those including shipping decisions.

It is worth mentioning Konur and Geunes (2012) that consider a location *game* between firms. While only one firm decides a location all in the above-mentioned research, all competing firms decide their own locations considering the resulting oligopolistic competition. Konur and Geunes (2012) consider spatially separated markets and shipping between markets and facilities in the presence of traffic congestion. They used inverse demand functions to model the market prices. Their model is not a Stackelberg model, as there is no leader. However, to find an equilibrium, they used a two-stage approach, where the first stage determines a location equilibrium and the second stage determines a production-shipping equilibrium. This is structurally similar to the Stackelberg-Cournot-Nash models in Table 1 in the sense that, for any given first stage result, *i.e.* locations, Konur and Geunes (2012) use a variational inequality model to determine the second stage solution. To determine the best response for the first stage location decision, they use a full enumeration for each firm.

As is in Tobin et al. (1995), Miller et al. (2007), and Konur and Geunes (2012), the full enumeration is a popular choice of methods to find an optimal location to place a new facility. When the number of all location candidates is small, full enumerations work fine. However, when there are many location candidates, a full enumeration would be computationally very heavy, because a variational inequality problem, whose size would be also very large, needs to be solved for each location candidate. In this paper, we attempt to reduce the number of variational inequality problems by search the set of locations by simulated annealing.

The remainder of this paper is organized as follows. In Section 2, we describe the Stackelberg-Cournot-Nash model we consider for locating a facility in the form of a mixed integer program with equilibrium constraints, where the lower-level equilibrium problem is formulated as a variational inequality. In Section 3, we first consider a branch-and-bound method for the KKT conditions of the lower-level equilibrium problem and present a full enumeration approach using a fixed-point algorithm for solving the lower-level variational inequality problem. In Section 4, we propose a simulated annealing algorithm and test it in a test network. In Section 5, we conclude this paper with some remarks.

2 The Stackelberg-Cournot-Nash Model

Let us begin by more formally describing the facility location problem described above. Suppose that some set $\mathcal{F}$ of firms operate manufacturing facilities located on a network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, where they manufacture some product which they sell in competition with one another in some set of markets $\mathcal{M} \subseteq \mathcal{N}$. Each firm must pay some per-unit manufacturing cost, as well as a transportation cost for each unit which is proportional to the distance that the unit is shipped from the facility where it is manufactured to the market where it is sold. The market price in each market varies inversely with the amount of product offered for sale in that market. Each firm seeks independently to maximize its own profit subject to these constraints. We will use the following definitions of mathematical notations:

- $\mathcal{N}_f$ is the set of nodes at which firm f has a manufacturing facility.
- q_i^f is the output of firm $f \in \mathcal{F}$ at its facility at $i \in \mathcal{N}_f$.
- Q_i^f is the maximum production capacity of firm f at its facility at i.
- $V_i^f(q_i^f)$ is the total cost of production for firm f at node i.
- s_{ij}^f is the amount shipped by firm f from its facility at node i to the market at node j.
- c_i^f is the allocation of the output of firm f to the market at node i.
- $\pi_i(c_i)$ is the price function of the market at node i, where $c_i = \sum_{f\in\mathcal{F}} c_i^f$.
- $\mathcal{W}_f$ is the set of OD pairs over which firm f ships goods.
- r_{ij} is the freight rate (tariff) charged per unit of flow s_{ij} for OD pair $(i,j) \in \mathcal{W}_f$.

2.1 The Problems of the Cournot Firms

Following the previous models, such as Tobin et al. (1995) and Miller et al. (1991), we assume that (1) $\pi_i(\cdot)$ is monotonically decreasing; (2) the revenue function $\pi_j(c_j)c_j^f$ is a concave function of c_j^f; and (3) $V_j^f(\cdot)$ is convex and monotonically increasing. In particular, we assume a simple linear inverse-demand function for each market j of the form

$$\pi_j(c_j) = a_j - b_j \left(\sum_{f\in\mathcal{F}} c_j^f \right) \tag{1}$$

and linear production costs

$$V_i^f(q_i^f) = K_i^f + d_i^f q_i^f \tag{2}$$

where the values a, b, d and K are constants.

Then firm $f \in \mathcal{F}$'s problem is:

$$\max J_f(c^f, q^f, s^f; c^{-f}) = \sum_{j \in \mathcal{M}} \pi_j(c_j) c_j^f - \sum_{i \in \mathcal{N}_f} V_i^f(q_i^f) - \sum_{(i,j) \in \mathcal{W}_f} r_{ij} s_{ij}^f \tag{3}$$

subject to

$$q_i^f - \sum_{j \in \mathcal{M}} s_{ij}^f \geq 0 \quad \forall i \in \mathcal{N}_f \tag{4}$$

$$\sum_{i \in \mathcal{N}_f} s_{ij}^f - c_j^f \geq 0 \quad \forall j \in \mathcal{M} \tag{5}$$

$$q_i^f \leq Q_i^f \quad \forall i \in \mathcal{N}_f \tag{6}$$

$$q_i^f \geq 0 \quad \forall i \in \mathcal{N}_f \tag{7}$$

$$c_j^f \geq 0 \quad \forall j \in \mathcal{M} \tag{8}$$

$$s_{ij}^f \geq 0 \quad \forall i \in \mathcal{N}_f, j \in \mathcal{M} \tag{9}$$

where constraints (4) ensure that firm f cannot ship from any site more than it manufactures there, and constraints (5) ensure that firm f cannot sell at any market more than it ships there. The result is a market equilibrium where each firm's manufacturing and distribution decisions are such that any deviation from them would result in decreased profits. We used the notation c^{-f} to denote the vector of allocation amounts to the markets of all other firms than f.

Before we proceed we define vector notations to simplify the mathematical expressions. We first denote the strategy vector of firm f by $x^f = (c^f, q^f, s^f)$ where $c^f = (c_j^f)_{j \in \mathcal{M}}$, $q^f = (q_i^f)_{i \in \mathcal{N}_f}$, and $s^f = (s_{ij}^f)_{(i,j) \in \mathcal{W}_f}$. We also define the feasible set of the firm f:

$$\Omega_f = \{x^f : (4) \text{ - } (9) \text{ hold}\}$$

We denote the strategies of all other firms than f by x^{-f}.

Under the assumptions on the functions, we can show that the objective function (3) is concave for any given c^{-f} (Facchinei and Pang, 2003, Section 1.4.3). Therefore, the first-order optimality condition for firm f's problem in variational inequality form is to find $x^f \in \Omega_f$ such that

$$[\nabla J_f(x^f)]^T(z^f - x^f) \leq 0 \quad \forall z^f \in \Omega_f$$

where z^f is a dummy vector of the same dimensionality as x^f.

Concatenating the above variational inequalities for all firm $f \in \mathcal{F}$, we obtain the following variational inequality formulation of the Nash equilibrium, as given by various authors, *e.g.* Harker and Pang (1990); Facchinei and Pang (2003): to find $x \in \Omega$ such that

$$\sum_{f \in \mathcal{F}} [\nabla J_f(x^f)]^T(z^f - x^f) \leq 0 \quad \forall z \in \Omega$$

where $x = (x^f)_{f \in \mathcal{F}}$ and $\Omega = \prod_{f \in \mathcal{F}} \Omega_f$. For more details of this equilibrium, including the existence and uniqueness results, see Tobin et al. (1995).

2.2 The Problem of the Stackelberg Firm

In this context, firm $g \in \mathcal{F}$ wishes to build an additional manufacturing facility. In order to determine the best possible location for the new facility (the choice which will maximize profits), it is necessary to find the choice which will result in the greatest profit for firm g at the subsequent market equilibrium. We define the additional binary variables

$$y_j = \begin{cases} 1 & \text{if firm } g \text{ builds its new facility at location } j \text{ or already has a facility there} \\ 0 & \text{otherwise} \end{cases}$$

using F_j to denote the fixed cost for firm g to build a new manufacturing facility at location j. In the case where firm g already has a facility at location j, assume $F_j = 0$. Then, because we are considering all possible locations as potential manufacturing sites for g, we may say that $\mathcal{N}_g = \mathcal{N}$, that Q_i^g and V_i^g are defined for all $i \in \mathcal{N}$ and that Ω_g will take the form

$$\Omega_g(y) = \{x^g : (10) - (15) \text{ hold}\}$$

where (10)-(15) are defined as follows (identically to (4) - (9) except for the fact that constraint (12) must ensure that firm g does not manufacture at any location where it does not have a facility):

$$q_i^g - \sum_{j\in\mathcal{M}} s_{ij}^g \geq 0 \quad \forall i \in \mathcal{N} \tag{10}$$

$$\sum_{i\in\mathcal{N}} s_{ij}^g - c_j^g \geq 0 \quad \forall j \in \mathcal{M} \tag{11}$$

$$q_i^g \leq Q_i^g y_i \quad \forall i \in \mathcal{N} \tag{12}$$

$$q_i^g \geq 0 \ \ \forall i \in \mathcal{N}_f \tag{13}$$

$$c_j^g \geq 0 \ \ \forall j \in \mathcal{M} \tag{14}$$

$$s_{ij}^g \geq 0 \quad \forall i \in \mathcal{N}_f, j \in \mathcal{M} \tag{15}$$

Then firm g's problem becomes

$$\begin{aligned} \max Z_g(x^g, y) &= \left[\sum_{j\in\mathcal{M}} \pi_j(c_j)c_j^g - \sum_{i\in\mathcal{N}} V_i^g(q_i^g) - \sum_{(i,j)\in\mathcal{W}_g} r_{ij}s_{ij}^g - \sum_{i\in\mathcal{N}} y_i F_i^g\right] \\ &= J_g(x^g; x^{-g}) - \sum_{i\in\mathcal{N}} y_i F_i^g \end{aligned} \tag{16}$$

subject to

$$\sum_{f\in\mathcal{F}} [\nabla J_f(x^f)]^T (z^f - x^f) \leq 0 \quad \forall z \in \Omega(y) \tag{17}$$

$$x \in \Omega(y) \tag{18}$$

$$y_i \in \{0, 1\} \quad \forall i \in \mathcal{N} \tag{19}$$

$$\sum_{i\in\mathcal{N}} y_i = 1 \tag{20}$$

which is a mixed-integer program with equilibrium constraints (17), where we used the following definition

$$\Omega(y) = \Omega_g(y) \times \prod_{f\in\mathcal{F}, f\neq g} \Omega_f$$

In the next section, we discuss methods to solve this problem.

3 Numerical Methods

A discretely-constrained mathematical program with equilibrium constraints is difficult to solve, due to convexity issues as well as the discrete constraints. Various approaches have been attempted; a brief review of these is given in Gabriel et al. (2010). We will begin by attempting one of these, namely branch and bound.

3.1 Branch and Bound

The variational inequality constraint (17) may be written as follows: to find $x \in \Omega$ such that

$$\sum_{f\in\mathcal{F}} [\nabla J_f(x^f)]^T z^f \leq \sum_{f\in\mathcal{F}} [\nabla J_f(x^f)]^T x^f \quad \forall z \in \Omega(y) \tag{21}$$

Suppose that we have a solution $x \in \Omega$ on hand. Then, the variational inequality problem (21) may be written of the following mathematical optimization problem form:

$$\max_z \sum_{f\in\mathcal{F}} [\nabla J_f(x^f)]^T z^f \tag{22}$$

subject to

$$h^f(z^f; y) \leq 0 \quad \forall f \in \mathcal{F} \tag{23}$$

where we expressed the inequalities in the set Ω_f by a vector of linear functions $h^f(\cdot; y)$ for each $f \in \mathcal{F}$ and any y. For the given x, (22) is a linear programming problem with the decision variable z. However, we know that a solution of (22) is $z^* = x$. Therefore for $z^* = x$

to be optimal to (22), x needs to satisfy the following Karush-Kuhn-Tucker conditions:

$$\nabla J_f(x^f; x^{-f}) + (\mu^f)^T \nabla h^f(x^f; y) = 0 \tag{24}$$

$$(\mu^f)^T h^f(x^f; y) = 0 \tag{25}$$

$$\mu^f \geq 0 \tag{26}$$

for all $f \in \mathcal{F}$, where μ^f is a vector of dual variables. This type of analysis is well explained in Friesz (2010). Note that the problem (22) is not intended to be used in any solution algorithm, but devised to help the derivation of the KKT conditions.

The resulting problem that is equivalent to the problem (16) is of the form:

$$\max Z_g(x^g, y) = J_g(x^g; x^{-g}) - \sum_{i \in \mathcal{N}} y_i F_i^g \tag{27}$$

subject to

$$\nabla J_f(x^f; x^{-f}) + (\mu^f)^T \nabla h^f(x^f; y) = 0 \quad \forall f \in \mathcal{F} \tag{28}$$

$$(\mu^f)^T h^f(x^f; y) = 0 \quad \forall f \in \mathcal{F} \tag{29}$$

$$\mu^f \geq 0 \quad \forall f \in \mathcal{F} \tag{30}$$

$$h^f(x^f; y) \leq 0 \quad \forall f \in \mathcal{F} \tag{31}$$

$$y_i \in \{0, 1\} \quad \forall i \in \mathcal{N} \tag{32}$$

$$\sum_{i \in \mathcal{N}} y_i = 1 \tag{33}$$

which may then be solved by a branch and bound approach. This problem has two challenging components: first is the binary variable y, and second is the complementarity condition (29). During the branch-and-bound processes, we relax constraints (32), replacing them with linear constraints of the form $0 \leq y_i \leq 1$, as well as the complementarity constraints (29). For a given node of the branch and bound tree, we solve the comparatively simple relaxed subproblem. The subproblem corresponding to a given node of the tree will either be infeasible, feasible to the unrelaxed problem, or feasible to the relaxed subproblem but violating some of the relaxed constraints. In the latter case, we may branch further by adding a constraint to enforce either the binarity of y_i for some i or the complementarity condition (29) for some f. At each iteration, we find the two leaf nodes which violate the relaxed constraints most strongly, i.e. the leaf node with the closest y_i to $\frac{1}{2}$, and the leaf node with the largest $\left|h_k^f \mu_k^f\right|$, where the subscript k is used to denote k-th element of each vector. We then branch on one of these two nodes (choosing randomly between the two). We continue branching until fathoming due to bound, infeasibility or optimality. See Edmunds and Bard (1992) for a more complete description of this procedure.

The disadvantage of the branch-and-bound method is that it becomes computationally intractable even for relatively small problem sizes. Furthermore, the "large" problems considered in Edmunds and Bard (1992) were still rather small ($\mid h \mid= 10$, whereas even a small problem of the sort we are considering will have hundreds if not thousands of such constraints). This leads us to consider an alternative solution procedure.

3.2 Full Enumeration Using Fixed-Point Iteration

A simpler approach is to do a full enumeration of the possible facility location choices, as similarly done in Tobin et al. (1995), Miller et al. (2007), and Konur and Geunes (2012). This will require us to solve for each possible value of y a variational inequality problem: to find $x \in \Omega$

$$\sum_{f \in \mathcal{F}} [\nabla J_f(x^f)]^T (z^f - x^f) \leq 0 \quad \forall z \in \Omega(y) \tag{34}$$

and call the solution x^*. Then we compare $Z_g(x^*, y) = J_g(x^*) - \sum_{i \in \mathcal{N}} y_i F_i^g$ to identify the optimal location decision y^* that achieves the maximum $Z_g(x^*, y^*)$.

The variational inequality problem (34) may be solved by a fixed-point iteration of the form

$$x^{f,(k+1)} = P_{\Omega_f(y)}[x^{f,(k)} + \alpha \nabla J_f(x^{(k)})]$$

problem size	Branch and Bound			Full Enumeration
(number of locations)	runtime (seconds)	nodes explored	solution quality	runtime (seconds)
5	19.56	1804	0.9899	7.61
10	49.17	3102	0.9883	15.98
20	172.32	5806	0.9936	38.49
50	3303.03	15459	0.2046	176.52
100	∞	-	-	1437.54

Table 2: Comparison of performance of branch and bound with full enumeration.

for all $f \in \mathcal{F}$ for some suitable step sizes $\alpha > 0$, where P_Ω is the operator of minimum-norm projection onto the feasible region Ω, *i.e.*

$$P_\Omega(x^k) = \arg\min_{x\in\Omega} \left|\left|x - x^k\right|\right|^2$$

Since we assumed linear inverse demand functions and linear productions costs, we can in fact build a convex optimization problem that is equivalent to (34) (Facchinei and Pang, 2003, Section 1.4.3). Instead of using the fixed-point iteration, we can use a commercial optimization solver to solve the equivalent optimization problem. However, we note that the fixed-point iteration for the variational inequality problem (34) is in fact same as the gradient projection algorithm for solving the equivalent optimization problem.

3.3 Tests on Randomly Generated Networks

The branch-and-bound and full enumeration algorithms were compared for a number of randomly generated sample problems. To generate problem instances, we first located N nodes randomly based on uniform distributions on a box-constrained space, and assumed a complete network, *i.e.*, all nodes are connected to each other. The distance between any two nodes is measured by the Euclidean distance based on the coordinates of the nodes. We used MATLAB 2010a on a generic personal computer running Windows 7 with 3.10GHz Intel Core i5-2400 CPU, and 4GB RAM. The all runtimes reported in this paper are based on runs on this computing environment. For the full enumeration, we used `tomlab/qp-minos` to solve

Nash equilibrium sub-problems in an equivalent optimization form, with margin of 10^{-6} for both feasibility and convergence.

For each problem size N, we solved 200 instances, and we report average values in Table 2. The branch and bound solution quality is the average ratio of the best solution found by branch and bound to that found by full enumeration (*i.e.* to the optimal solution). The very poor average solution quality for branch and bound with $N = 50$ reflects the fact that on many occasions for $N = 50$ the branch and bound algorithm terminated due to time constraints without finding any feasible solution at all. For $N = 100$ branch and bound was not attempted due to excessively high running times. The runtimes for the full enumeration method are reported in Table 2, and they are faster than the branch-and-bound method, in all problem instances.

4 A Heuristic Search Algorithm by Simulated Annealing

As seen in Table 2, it becomes expensive to solve the problem by full enumeration, even for fairly modest-sized cases. Therefore it will be useful to find a heuristic which can find a good quality solution when evaluating a limited number of candidate solutions. Also, as the cost of evaluating any given candidate solution is relatively low, but the number of candidate solutions may be large, we would benefit from some way of limiting or directing our search of the solution space. A number of types of heuristics for limiting exploration of the solution space exist. One such is genetic algorithms. However, it is unclear how a genetic algorithm could be adapted to the problem at hand. The obvious genetic encoding for our problem would be to use the vector y as the genome. For example, since we locate only one new facility, all genomes for feasible solutions are an equal distance away from each other, which means that the genetic algorithm can get no purchase on the problem; it is essentially the same thing as a random search.

Simulated annealing, by contrast, may be more naturally applied to the problem at hand. Simulated annealing already has a history of use in location problems, see for example Golden and Skiscim (1986), Murray and Church (1996), Drezner et al. (2002), Arostegui et al. (2006), Redondo et al. (2009), and Parvaresh et al. (2013). Simulated annealing requires only a notion of adjacency. The solution space is searched by starting with some arbitrary incumbent (or perhaps multiple incumbents if a multi-start heuristic is used) and then evaluating adjacent candidates. If some criterion is met, then the candidate solution replaces the current incumbent. Because any reasonable definition of adjacency may be used, simulated annealing may therefore be adapted so as to solve our problem.

In our case if we are looking for the optimal site for a single new facility our candidate solutions have the property that each corresponds to some point in space. Therefore we can define adjacency for two candidate solutions as some function of the relative positions of the corresponding locations. If the various sites have truly arbitrary production costs (*i.e.* costs vary erratically from site to site and the costs of nearby sites are not strongly correlated) such a search may not be practical. If costs are "reasonable" enough, the objective function of a candidate solution will be positively correlated with those of its closest geographical neighbors.[1] This may be a tenable assumption especially if transportation costs are a significant enough portion of total cost, and are directly proportional to the distance between the production facility and the market being served. A simulated annealing-based algorithm for our problem would then be approximately as follows. Given an incumbent solution, choose as our next candidate solution one of the closest as yet unexamined locations to the incumbent. If the objective function of the candidate is better than the incumbent, replace the incumbent. Otherwise replace it probabilistically as determined by a cooling function of the form

$$c(Z_{\text{cand}}, t) = \exp\left(\frac{t}{500 \mid \mathcal{N} \mid}(Z_{\text{cand}} - Z_{\text{inc}})\right)$$

where Z_{cand} is the equilibrium profit to firm g given the candidate solution, Z_{inc} is the equilibrium profit to firm g given the incumbent solution, and t is the time since the beginning of the algorithm. The algorithm is thus as follows:

Step 0: Let $t \leftarrow 1$. For each $n \in \mathcal{N}$, compute its set of neighbors $\mathcal{A}(n)$ as the subset of

[1]This discussion assumes $p = 1$ (or that the number of "movable" sites is 1) but the same concept applies if $p > 1$ if we assume that two candidate solutions are adjacent if they have $p - 1$ facilities in common and that the remaining facility is adjacent in the sense that it would be for $p = 1$.

points in $\mathcal{N}$ which are less than some threshold distance from n.

Step 1: Pick $i \in \mathcal{N}$ at random as the incumbent (choosing from all nodes of $\mathcal{N}$ with equal probability). Solve the corresponding variational inequality (34) to find the resulting market equilibrium, either using fixed point iteration or a commercial solver for the equivalent optimization formulation, assuming that firm g's new facility is placed in location i, and compute Z_{inc}, the resulting profit for firm g.

Step 2: Pick candidate location j, choosing from all unvisited neighbors of the incumbent with equal probability. If all neighbors of the incumbent have already been visited, choose any unvisited location. Evaluate the objective function Z_{cand}.

Step 3: Evaluate the cooling function $c(Z_{\text{cand}}, t)$ and choose a random number $X \sim U[0, 1]$. If $X < c(Z_{\text{cand}}, t)$, make j the incumbent and set $Z_{\text{inc}} \leftarrow Z_{\text{cand}}$. (Note that this will always be the case if $Z_{\text{cand}} > Z_{\text{inc}}$.)

Step 4: If the desired number of iterations has been reached, terminate. Else $t \leftarrow t + 1$ and go to step 2.

The optimal average size for $\mathcal{A}(n)$ was determined by trial and error, as shown in Figure 1. Figure 2 shows a comparison between full enumeration using a random search of the solution space and using the simulated annealing heuristic. At first, random search performs better, in the sense that after examining only a few candidate solutions the best one found so far is likely to be better. This is because if the simulated annealing procedure happens to start in a region where candidate locations are poor, it must move gradually to an area with better performing solutions. Random search is under no such constraint. However, simulated annealing eventually surpasses random search and almost always finds a good quality solution after having examined only a fraction of the solution space.

Observing this behavior, we make a simple modification to the heuristic. Rather than pick a single random starting point for the heuristic, we evaluate several potential starting points and start at the point with the best objective function value. The results after making this modification to the procedure are shown in Figure 3. This simulated annealing heuristic

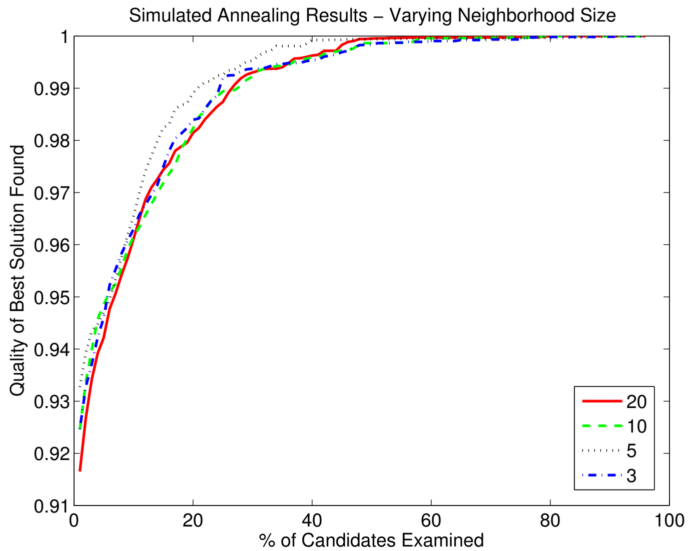

Figure 1: Comparison of simulated annealing with different neighborhood sizes, with $N = 100$. The neighbors of node i are defined as being all nodes within some radius of i. The best results are seen when the radius is chosen so as to make each node have an average of about 5 neighbors.

performs equally well to random search at the beginning (indeed it starts out by doing a random search), but then proceeds to outperform random search thereafter.

The computational time and the solution quality of simulated annealing depend on when we terminate the algorithm. When we apply simulated annealing, we cannot use the quality of best solution found as in the vertical axis of Figures 1–3, since we do not know the optimal solution a priori. However, in all cases, simulated annealing seems producing a solution with the optimality gap less than 1%, after it explores about 30% of all location choices. Therefore, one may use 30% as a guideline for stopping criteria. In this case, the computational time for simulated annealing would be around 30% of the computational time of the full enumeration method. For the $N = 100$ example, it is 30% of 1437.54 seconds, which is about 431 seconds. The computational environments for simulated annealing were same as for the full enumeration method, already reported in Section 3.3.

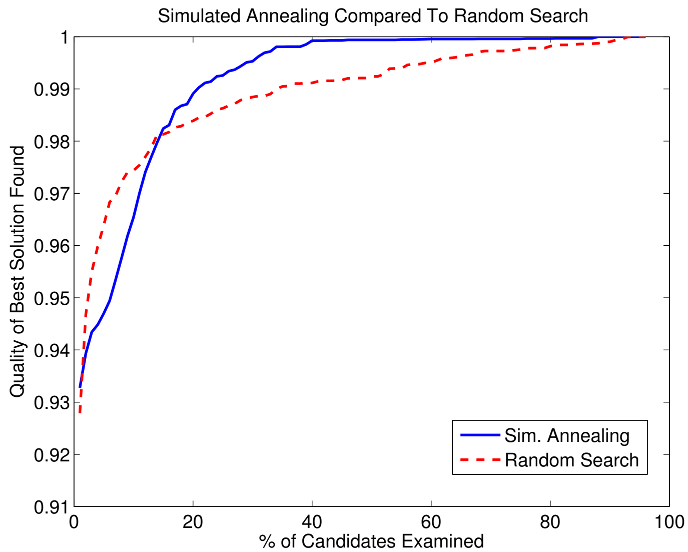

Figure 2: Comparison of simulated annealing and random search. Shown is the quality of the best solution found, as a function of the percentage of the solution space yet explored, with $N = 100$. Simulated annealing essentially always found optimal or near optimal solution after exploring about a third of the solution space.

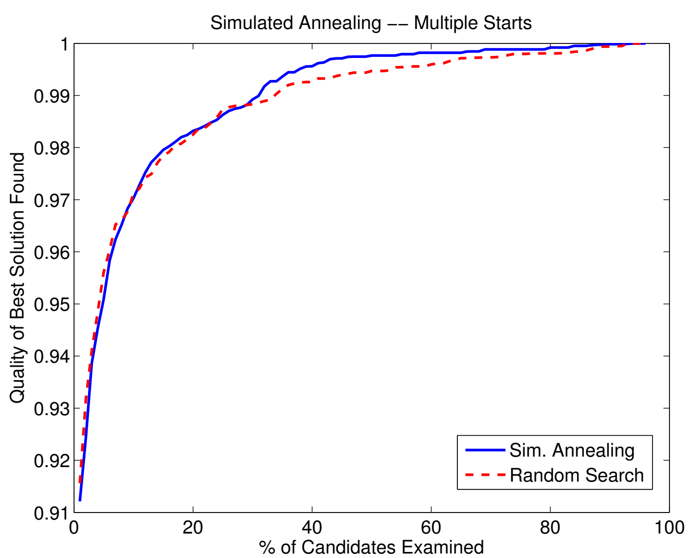

Figure 3: Simulated annealing with multiple starting points. (Multiple starting points are evaluated and the procedure is started from the point with the best objective function value.) Shown is the quality of the best solution found, as a function of the percentage of the solution space yet explored, with $N = 100$.

5 Concluding Remarks

We have presented a heuristic for solving a manufacturing site location problem. Preliminary results suggest that it is capable of finding good solutions to problems of moderate size ($| \mathcal{N} | \approx 100$ or somewhat larger). Furthermore, it has the advantage that fairly good quality solutions are found quickly, so the entire solution space does not necessarily need to be explored. However it should be noted that it too will have difficulty with larger problems. As $| \mathcal{F} |$ and $| \mathcal{M} |$ grow, the projection problem P_Ω becomes more expensive and convergence behavior of the fixed point iteration may deteriorate, making it more difficult to explore the solution space.

There other possible approaches which could be investigated. One such would be a modified definition of adjacency in the simulated annealing procedure. In particular, we could define the n closest points *in each of several directions* as adjacent, rather than merely the n closest points over all, as done in the current implementation. Another possibility would be rather than picking one adjacent node at random to evaluate, a number of adjacent nodes could be examined, and the most improving one per distance away from the incumbent would be chosen. Note that this is not simulated annealing but more of a primitive kind of gradient search.

Possible future extensions may be competitive location problems in consideration of robust decision-making under uncertainty (Berglund and Kwon, 2013) and multi-echelon supply chain network involving hubs (Shahabi et al., 2013).

References

Arostegui, M., S. Kadipasaoglu, and B. Khumawala (2006). An empirical comparison of tabu search, simulated annealing and genetic algorithms for facilities location problems. *International Journal of Production Economics 103*(2), 742–754.

Berglund, P. and C. Kwon (2013). Robust facility location problem for hazardous waste transportation. *Networks and Spatial Economics (forthcoming).*

Cournot, A. (1838). *Recherches sur les principes mathématiques de la théorie des richesses.* L. Hachette.

Dafermos, S. and A. Nagurney (1984). Sensitivity analysis for the general spatial economic equilibrium problem. *Operations Research 32*(5), 1069–1086.

Dobson, G. and U. Kamarkar (1985). Competitive location on a network. *Operations Research 35*(4), 565–574.

Drezner, T., Z. Drezner, and S. Salhi (2002). Solving the multiple competitive facilities location problem. *European Journal of Operational Research 142*(1), 138–151.

Edmunds, T. and J. Bard (1992). An algorithm for the mixed-integer nonlinear bilevel programming problem. *Annals of Operations Research 34*(1), 149–162.

Eiselt, H., G. Laporte, and J. Thisse (1993). Competitive location models: A framework and bibliography. *Transportation Science 27*(1), 44–54.

Facchinei, F. and J.-S. Pang (2003). *Finite-Dimensional Variational Inequalities and Complementarity Problems.* Springer-Verlag.

Friesz, T. (2010). *Dynamic Optimization and Differential Games.* Springer.

Friesz, T., P. Harker, and R. Tobin (1984). Alternative algorithms for the general network spatial price equilibrium problem. *Journal of Regional Science 24*(4), 475–507.

Friesz, T., T. Miller, and R. Tobin (1988). Competitive network facility location models: A survey. *Papers of the Regional Science Association 65*(1), 47–57.

Friesz, T., M. Rigdon, and R. Mookherjee (2006). Differential variational inequalities and shipper dynamic oligopolistic network competition. *Transportation Research Part B 40*(6), 480–503.

Friesz, T., R. Tobin, and T. Miller (1989). Existence theory for spatially competitive network facility location models. *Annals of Operations Research 18*(1), 267–276.

Gabriel, S., Y. Shim, A. Conejo, S. de la Torre, and S. Garcia-Bertrand (2010). A Benders decomposition method for discretely-constrained mathematical programs with equilibrium constraints. *Journal of the Operational Research Society 61*(9), 1404–1419.

Golden, B. and C. Skiscim (1986). Using simulated annealing to solve routing and location problems. *Naval Research Logistics Quarterly 33*(2), 261–279.

Hale, T. S. and C. R. Moberg (2003). Location science research: A review. *Annals of Operations Research 123*(1-4), 21–35.

Harker, P. (1984). A variational inequality approach for the determination of oligopolistic market equilibrium. *Mathematical Programming 30*(1), 105–111.

Harker, P. (1986). Alternative models of spatial competition. *Operations Research 34*(3), 410–425.

Harker, P. T. and J.-S. Pang (1990). Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications. *Mathematical Programming 48*(1-3), 161–220.

Konur, D. and J. Geunes (2012). Competitive multi-facility location games with non-identical firms and convex traffic congestion costs. *Transportation Research Part E: Logistics and Transportation Review 48*(1), 373–385.

Kress, D. and E. Pesch (2011). Sequential competitive location on networks. *European Journal of Operational Research inpress*, doi:10.1016/j.ejor.2011.06.036.

Luo, Z., J. Pang, and D. Ralph (1996). *Mathematical programs with equilibrium constraints*. Cambridge University Press.

Miller, T., T. Friesz, and R. Tobin (1992). Heuristic algorithms for delivered price spatially competitive network facility location problems. *Annals of Operations Research 34*(1), 177–202.

Miller, T., T. Friesz, R. Tobin, and C. Kwon (2007). Reaction function based dynamic location modeling in Stackelberg-Nash-Cournot competition. *Networks and Spatial Economics 7*(1), 77–97.

Miller, T., R. Tobin, and T. Friesz (1991). Stackelberg games on a network with cournot-nash oligopolistic competitors. *Journal of Regional Science 31*(4), 435–454.

Miller, T. C., T. L. Friesz, and R. L. Tobin (1996). *Equilibrium Facility Location on Networks*. Springer-Verlag, Berlin, Heidelberg, New York.

Murray, A. and R. Church (1996). Applying simulated annealing to location planning models. *Journal of Heuristics 2*(1), 31–53.

Nagurney, A. (1987). Computational comparisons of spatial price equilibrium methods. *Journal of Regional Science 27*(1), 55–76.

Nagurney, A. (1998). *Network economics: A variational inequality approach* (Revised Second Edition ed.). Springer.

Parvaresh, F., S. H. Golpayegany, S. M. Husseini, and B. Karimi (2013). Solving the p-hub median problem under intentional disruptions using simulated annealing. *Networks and Spatial Economics (forthcoming)*.

Plastria, F. (2001). Static competitive facility location: An overview of optimisation approaches. *European Journal of Operational Research 129*(3), 461–470.

Plastria, F. and L. Vanhaverbeke (2007). Aggregation without loss of optimality in competitive location models. *Networks and Spatial Economics 7*(1), 3–18.

Redondo, J. L., J. Fernández, I. García, and P. M. Ortigosa (2009). Solving the multiple competitive facilities location and design problem on the plane. *Evolutionary Computation 17*(1), 21–53.

Samuelson, P. (1952). Spatial price equilibrium and linear programming. *The American Economic Review 42*(3), 283–303.

Santos-Peñate, D. R., R. Suárez-Vega, and P. Dorta-González (2007). The leader-follower location model. *Networks and Spatial Economics 7*(1), 45–61.

Serra, D. and C. ReVelle (1999). Competitive location and pricing on networks. *Geographical Analysis 31*(2), 109–129.

Shahabi, M., S. Akbarinasaji, A. Unnikrishnan, and R. James (2013). Integrated inventory control and facility location decisions in a multi-echelon supply chain network with hubs. *Networks and Spatial Economics (forthcoming).*

Sherali, H., A. L. Soyster, and F. H. Murphy (1983). Stackelberg-Nash-Cournot equilibria: Characterizations and computations. *Operations Research 31*(2), 253–276.

Takayama, T. and G. G. Judge (1971). *Spatial and Temporal Price and Allocation Models.* Amsterdam: North-Holland.

Tobin, R. and T. Friesz (1986). Spatial competition facility location models: Definition, formulation and solution approach. *Annals of Operations Research 6*(3), 47–74.

Tobin, R., T. Miller, and T. Friesz (1995). Incorporating competitors' reactions in facility location decisions: A market equilibrium approach. *Location Science 3*(4), 239–253.