Worst-case Conditional Value-at-Risk Minimization for Hazardous Materials Transportation

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Despite significant advances in risk management, the routing of hazardous materials (hazmat) has relied on relatively simplistic methods. In this paper, we apply an advanced risk measure, called conditional value-at-risk (CVaR), for routing hazmat trucks. CVaR offers a flexible, risk-averse, and computationally tractable routing method that is appropriate for hazmat accident mitigation strategies. The two important data types in hazmat transportation are accident probabilities and accident consequences, both of which are subject to many ambiguous factors. In addition, historical data are usually insufficient to construct a probability distribution of accident probabilities and consequences. This motivates our development of a new robust optimization approach for considering the worst-case CVaR (WCVaR) under data uncertainty. We study important axioms to ensure that both the CVaR and WCVaR risk measures are coherent and appropriate in the context of hazmat transportation. We also devise a computational method for WCVaR and demonstrate the proposed WCVaR concept with a case study in a realistic road network.

Key words: hazardous materials transportation; conditional value-at-risk; robust optimization

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1. Introduction

The Pipeline and Hazardous Materials Safety Administration (2013) defines hazardous materials (hazmat) as "a substance or material capable of posing an unreasonable risk to health, safety, or property when transported in commerce," and the US Department of Transportation manages hazmat transportation with nine classifications (Federal Motor Carrier Safety Administration 2001). Hazmat accidents can result in significant injuries to the population and damage to the environment. In 2007, hazmat shipments accounted for about 18 percent of the total freight shipped in the U.S. on a tonnage basis (U.S. Census Bureau 2007).

Trucks are the most widely used mode for hazmat transportation over relatively short distances. The popularity of trucks for short-haul hazmat transport stems from their flexibility of operation, i.e., their ability to pick up and drop off hazmat close to its point of origin and destination, respectively. As might be expected, the number of hazmat incidents involving truck transport is also the highest of the transport modes. While all transport modes can cause severe consequences to the community, trucks have a more direct impact on public safety. People can be seriously injured or killed, important infrastructure systems can be damaged, and residential environments can be destroyed.

In hazmat transportation, the extreme consequences of accidents must be avoided. However, current routing methods rely on simplistic rules or the most economical route, despite the significant advances being made in the field of risk management. In addition, some of the existing routing methods reported in the literature do not provide *risk-averse* routing, and others lack the *flexibility* necessary to accommodate various practical factors. With the objective of providing a simultaneously risk-averse and flexible route decision-making process, we study new routing methods based on the concept of conditional value-at-risk (CVaR).

The CVaR concept is closely related to value-at-risk (VaR). The VaR concept has been widely applied in the financial and economic fields (Duffie and Pan 1997, Linsmeier and Pearson 2000) and its application to hazmat transportation was recently introduced (Kang et al. 2014a,b). Critics claim that, as a risk measure, VaR ignores and cuts off what could happen in the distributions tail; hence some argue that VaR can bring inaccurate risk perception to decision makers which could lead to catastrophic outcomes (Einhorn 2008, Nocera 2009). This has motivated the consideration of CVaR in hazmat transportation (Toumazis et al. 2013, Toumazis and Kwon 2013).

Assuming that data are uncertain within given sets, this paper considers a worst-case CVaR (WCVaR) minimization problem for hazmat routing. While CVaR has the potential to be an important risk measure and decision-making tool in the context of hazmat transportation of a low-probability-high-consequence nature, there is a critical issue that must be addressed in any hazmat routing method—*data uncertainty*. While the consequences of hazmat accidents usually cause serious problems, hazmat-accident data are usually insufficient for constructing probability distributions. The accident probability of hazmat trucks in each road segment is difficult to estimate and consequences from such accidents are dependent on various and uncertain factors such as the severity of the accident, local population density and weather conditions at the time of the accident, the type of hazmat shipment, and the quantity of hazmat released (Kwon et al. 2013). These factors make any hazmat routing method based on the stochasticity of data less meaningful.

We distinguish the two kinds of robustness: one derived from the CVaR concept itself, and the other from the worst-case consideration in WCVaR. CVaR, especially with a high probability threshold, provides protections against high levels of loss in the underlying risk, therefore providing robustness. In the context of hazmat transportation, when the probability threshold is very high, the robustness of the CVaR concept is related to the worst accident location. On the other hand, the worst-case consideration in WCVaR provides protections against data inaccuracy, and its robustness is related to the worst realizations of accident probabilities and accident consequences. Therefore,

Table 1 Mathematical Notation

$G(\mathcal{N},\mathcal{A})$	A graph of transportation network
$\mathcal{N}$	Set of nodes, $\lvert\mathcal{N}\rvert = n$
$\mathcal{A}$	Set of arcs, $\lvert\mathcal{A}\rvert = m$
p_{ij}	Accident probability along arc (i,j)
c_{ij}	Accident consequence along arc (i,j)
$\mathcal{P}$	Set of all available paths for given O-D pair
$\mathcal{C}$	Set of ascending-order sorted arc consequences in G
$\mathcal{A}^l$	Arc set for path l, and $\lvert\mathcal{A}^l\rvert$ is the number of arcs in l
$\mathcal{C}^l$	Set of ascending-order sorted arc consequences for path l
R^l	Discrete random variable for the risk along path $l \in \mathcal{P}$

these two kinds of robustness are fundamentally different. Furthermore, the uncertainty regarding the accident locations can be represented as a probability distribution, while the uncertainty of data involves intervals only.

The contributions of this paper are summarized as follows: First, we formally introduce WCVaR in hazmat routing. Second, we provide axiomatic studies to the validity of CVaR and WCVaR in the context of hazmat transportation and the meanings of the axioms proposed by Artzner et al. (1999) and Erkut and Verter (1998). Third, we devise an efficient computational method for solving WCVaR minimization problems for hazmat routing. Fourth, we confirm the validity of the WCVaR concept and the practicability of the proposed algorithm with a case study in a realistic road network.

2. Related Models

In this section, we provide a brief summary of existing approaches in hazmat routing. Let us consider a transportation network $G = (\mathcal{N},\mathcal{A})$, where $\mathcal{N}$ is the set of nodes and $\mathcal{A}$ the set of arcs. Each arc $(i,j) \in \mathcal{A}$ is assigned two attributes: accident probability p_{ij} and accident consequence c_{ij}. Values of c_{ij} can be determined by a risk assessment method, for example, the λ-neighborhood concept proposed by Batta and Chiu (1988), and values of p_{ij} should be collected from certain data sources. All the mathematical notation used in this Section is provided in Table 1. Assume path l is an ordered set of arcs $\mathcal{A}^l = \{(i_k, j_k) \in \mathcal{A} : k = 1, 2, \ldots, |\mathcal{A}^l|\}$ where (i_k, j_k) is the k-th arc in the path. To measure the risk that this path generates, the Traditional Risk method (TR) employs the expected value of the consequence along path l (Sherali et al. 1997, Erkut and Verter 1998):

$$\mathbb{E}\left[R^l\right] = \sum_{(i_k,j_k)\in\mathcal{A}^l} \prod_{(i_h,j_h)\in\mathcal{A}^l, h<k} (1 - p_{i_h j_h}) p_{i_k j_k} c_{i_k j_k} \tag{1}$$

The risk R^l along the path l has the following discrete distribution:

$$\Pr\{R^l = x\} = \begin{cases} 1 - \displaystyle\sum_{(i_k,j_k)\in\mathcal{A}^l} \prod_{(i_h,j_h)\in\mathcal{A}^l, h<k} (1 - p_{i_h j_h}) p_{i_k j_k} & \text{if } x = 0 \\ \displaystyle\prod_{(i_h,j_h)\in\mathcal{A}^l, h<k} (1 - p_{i_h j_h}) p_{i_k j_k} & \text{if } x = c_{i_k j_k} \quad \forall (i_k, j_k) \in \mathcal{A}^l \end{cases} \tag{2}$$

Table 2 Classic Models (Erkut and Ingolfsson 2005)

Model	Risk Measure	Objective	Model	Risk Measure	Objective
TR	Expected Risk	$\min_{l\in\mathcal{P}} \sum_{(i,j)\in\mathcal{A}^l} p_{ij}c_{ij}$	MM	Maximum Risk	$\min_{l\in\mathcal{P}} \max_{(i,j)\in\mathcal{A}^l} c_{ij}$
PE	Population Exposure	$\min_{l\in\mathcal{P}} \sum_{(i,j)\in\mathcal{A}^l} c_{ij}$	MV	Mean-Variance	$\min_{l\in\mathcal{P}} \sum_{(i,j)\in\mathcal{A}^l} (p_{ij}c_{ij} + kp_{ij}(c_{ij})^2)$
IP	Incident Probability	$\min_{l\in\mathcal{P}} \sum_{(i,j)\in\mathcal{A}^l} p_{ij}$	DU	Disutility	$\min_{l\in\mathcal{P}} \sum_{(i,j)\in\mathcal{A}^l} p_{ij}(\exp(kc_{ij}-1))$
PR	Perceived Risk	$\min_{l\in\mathcal{P}} \sum_{(i,j)\in\mathcal{A}^l} p_{ij}(c_{ij})^q$	CR	Conditional Probability	$\min_{l\in\mathcal{P}} \left(\sum_{(i,j)\in\mathcal{A}^l} p_{ij}c_{ij} \Big/ \sum_{(i,j)\in\mathcal{A}^l} p_{ij} \right)$

where indices k and h being integers between 1 and $|\mathcal{A}^l|$. Following Jin and Batta (1997), the distribution (2) can be approximated by

$$\Pr\{R^l = x\} \approx \begin{cases} 1 - \sum_{(i,j)\in\mathcal{A}^l} p_{ij} & \text{if } x = 0 \\ p_{ij} & \text{if } x = c_{ij} \quad \forall (i,j)\in\mathcal{A}^l \end{cases} \tag{3}$$

Erkut and Verter (1998) verified that approximation (3) introduced insignificant errors. All the subsequent discussions in this paper will use this approximated risk distribution.

With approximation (3), the TR model (1) can be written as $\mathbb{E}[R^l] \approx \sum_{(i,j)\in\mathcal{A}^l} p_{ij}c_{ij}$ that is much easier to optimize than (1), because the resultant problem is a shortest-path problem in which the product $p_{ij}c_{ij}$ represents the cost of traversing an arc (i,j). Various other models (Table 2) have been developed, focusing either only on one of the two attributes or on both. We refer the reader to the work of Erkut and Ingolfsson (2005) for a detailed review of these models.

Bell (2007) proposes a mixed-route model considering data uncertainty; however, it is limited to accident probabilities and requires path-enumeration for application. Other works address time-dependent hazmat transportation (Miller-Hooks and Mahmassani 1998, Chang et al. 2005, Androutsopoulos and Zografos 2012, Toumazis and Kwon 2013), equitable risk routes (Carotenuto et al. 2007), and emergency response decisions (Zografos and Androutsopoulos 2008), with consideration of traditional-risk minimization approaches. In addition, Waller and Ziliaskopoulos (2006) have applied chance-constrained optimization methods to traffic assignment problems to consider travel-demand uncertainty.

Compared with other risk measures used in hazmat transportation, we claim that CVaR is a better risk measure in the sense that it offers a control parameter—probability threshold or confidence level—that is easier to understand for flexible decision making, and it provides a risk-averse routing method. In addition, complicated CVaR problems can be solved efficiently.

In the context of traffic assignment, Chen and Zhou (2010) propose the concept of mean-excess travel time (METT) similar to CVaR. METT models consider the travelers' average travel time that exceeds a specified travel time budget in user equilibrium frameworks. Although under a different name, the notion of METT is consistent with CVaR. Wu and Nie (2011) compare METT and other risk models that consider travel time reliability using stochastic dominance. While a continuous distribution is assumed for travel time, a discrete distribution is used in hazmat transportation risk. This distinction enables us to devise a more efficient computational method for CVaR than METT.

Assuming that data are uncertain within given sets, this paper considers a worst-case CVaR (WCVaR) minimization problem for hazmat routing. For financial portfolio optimization applications, worst-cases have been considered for VaR (El Ghaoui et al. 2003) and CVaR (Zhu and Fukushima 2009). Our worst-case problem is structurally different from these problems. There are two types of uncertain data in hazmat problems: accident probability and accident consequence. Since the sources for these two types of data are generally different, the level of uncertainty in each data type is independent from each other and it is difficult to be considered as a single uncertain data (Kwon et al. 2013). We base our methodological developments on the robust shortest path problem with two uncertain multiplicative cost coefficients studied in Kwon et al. (2013).

3. Axiomatic Study of Conditional Value-at-Risk

Approximately—or exactly when the risk is a continuous random variable—CVaR is the expected value of risk that is greater than or equals to the VaR value at the given confidence level α; hence CVaR quantifies the risk in the long tail. We refer readers to Sarykalin et al. (2008) and Toumazis et al. (2013) for details. To understand the CVaR concept better in the context of hazmat transportation, this section provides an axiomatic study. To define CVaR, let us consider the following auxiliary function (Rockafellar and Uryasev 2000, Pflug 2000) with the approximation (3):

$$\Phi^l_\alpha(r) = r + \frac{1}{1-\alpha}\mathbb{E}[R^l - r]^+ = r + \frac{1}{1-\alpha}\left\{\left(1 - \sum_{(i,j)\in\mathcal{A}^l} p_{ij}\right)[0-r]^+ + \sum_{(i,j)\in\mathcal{A}^l} p_{ij}[c_{ij}-r]^+\right\} \tag{4}$$

where $[x]^+ = \max(x,0)$. The first term r is related to the VaR value, and the second term $\frac{1}{1-\alpha}\mathbb{E}[R^l - r]^+$ is related to the expected additional risk beyond VaR. Toumazis et al. (2013) show that the CVaR for path l can be measured by

$$\mathsf{CVaR}^l_\alpha \approx \min_{r\in\mathbb{R}} \Phi^l_\alpha(r) = \min_{r\in\mathbb{R}^+}\left[r + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^l} p_{ij}[c_{ij}-r]^+\right] \tag{5}$$

Finally the CVaR minimization is equivalent to minimizing Φ^l_α by choosing a path $l\in\mathcal{P}$ at the confidence level α. That is,

$$\min_{l\in\mathcal{P}} \mathsf{CVaR}^l_\alpha \approx \min_{l\in\mathcal{P}, r\in\mathbb{R}^+} \Phi^l_\alpha(r) = \min_{l\in\mathcal{P}, r\in\mathbb{R}^+}\left[r + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^l} p_{ij}[c_{ij}-r]^+\right] \tag{6}$$

which can be solved by a finite number of shortest path problems (Toumazis et al. 2013), mainly due to the discreteness of the risk distribution.

The CVaR is known to satisfy the four axioms by Artzner et al. (1999) to be a *coherent* risk measure for general loss distributions (Rockafellar and Uryasev 2002). The four axioms for a risk measure ξ that maps a random loss X to a real number are:

CA1 (Translation Invariance) For any real number m, $\xi(X+m)=\xi(X)+m$.

CA2 (Subadditivity) For all X_1 and X_2, $\xi(X_1+X_2)\le\xi(X_1)+\xi(X_2)$.

CA3 (Positive Homogeneity) For all $\lambda\ge 0$, $\xi(\lambda X)=\lambda\xi(X)$.

CA4 (Monotonicity) For all X_1 and X_2 with $X_1\le X_2$ a.s., $\xi(X_1)\le\xi(X_2)$.

The question is if the *approximated* version of CVaR in (5) is still coherent. Note that the approximated distribution (3) constitutes a complete probability distribution, and the approximated CVaR in (5) is the exact CVaR for the distribution (3). Since CVaR is coherent for the distribution (3), the approximated CVaR measure in (5) is also coherent.

While CVaR in hazmat transportation—both exact and approximated—is a coherent risk measure, we find that the translation invariance axiom **CA1** is less meaningful and the monotonicity axiom **CA4** is not applicable in the context of hazmat transportation. The meaning of $X+m$ in **CA1** in hazmat transportation is that the accident consequence increases by m, even when there is no accident. Although **CA1** is mathematically correct and valid, it reads improperly in the context of hazmat transportation. We also note that the condition in the monotonicity axiom **CA4** is not applicable to hazmat routing, as the axiom considers the case when the relationship between two random variables is that $R^{l_1}\le R^{l_2}$ a.s., i.e. $\Pr[R^{l_1}>R^{l_2}]=0$. In hazmat routing, $\Pr[R^{l_1}>R^{l_2}]$ is never zero, because it is possible that $R^{l_2}=0$ when $R^{l_1}>0$ as long as $p_{ij}>0$ for some $(i,j)\in\mathcal{A}^{l_1}$.

While the meaning of the positive homogeneity axiom **CA3** is clear in hazmat transportation, the meaning of the subadditivity axiom **CA2** is ambiguous. We provide a "translation" of **CA2** in the context of hazmat transportation with the approximate CVaR measure (5). Artzner et al. (1999) states the meaning of **CA2** as "a merger does not create extra risk". In the context of hazmat, we may similarly state as follows: using two paths by a same carrier does not create extra risk. We consider two such cases. First, when two different trucks from the same carrier use two different paths, say l_1 and l_2. Then X_1+X_2 in **CA2** requires a joint probability of $R^{l_1}+R^{l_2}$ where each is of the form (2). However, we can approximately write:

$$\Pr\{R^{l_1}+R^{l_2}=z\}\approx\begin{cases}1-\displaystyle\sum_{(i,j)\in\mathcal{A}^{l_1}\cup\mathcal{A}^{l_2}}p_{ij} & \text{if } z=0\\ p_{ij} & \text{if } z=c_{ij}\quad\forall(i,j)\in\mathcal{A}^{l_1}\cup\mathcal{A}^{l_2}\end{cases}\tag{7}$$

Note that $\Pr\{R^{l_1}+R^{l_2}=c_{ij}+c_{i'j'}\}\approx 0$ for any $(i,j)\in\mathcal{A}^{l_1}$ and $(i',j')\in\mathcal{A}^{l_2}$, which means that the probability of one accident along path l_1 and another accident along path l_2 at the same time is very small. Alternatively, we can consider a truck traveling both paths l_1 and l_2 within one trip, when l_2 starts where l_1 ends. In both cases, the translations of **CA2** are same with the approximation. The translation follows.

CA2' (Subadditivity) For all paths l_1 and l_2,

$$\min_{r\in\mathbb{R}^+}\left[r+\frac{1}{1-\alpha}\left\{\sum_{(i,j)\in\mathcal{A}^{l_1}} p_{ij}[c_{ij}-r]^+ + \sum_{(i,j)\in\mathcal{A}^{l_2}} p_{ij}[c_{ij}-r]^+\right\}\right] \leq$$

$$\min_{s\in\mathbb{R}^+}\left[s+\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_1}} p_{ij}[c_{ij}-s]^+\right] + \min_{t\in\mathbb{R}^+}\left[t+\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_2}} p_{ij}[c_{ij}-t]^+\right]$$

We state our finding as a proposition:

PROPOSITION 1. *The approximated CVaR in* (5) *for hazmat routing is a coherent risk measure in the sense of Artzner et al. (1999) satisfying* ***CA1****,* ***CA2'****, and* ***CA3****.*

All the proofs of propositions and lemmas are provided in the appendix.

Erkut and Verter (1998) proposed three axioms that path evaluation models in hazmat routing should satisfy. Let us denote a risk measure of path l by $\zeta^l(p,c)$, where p is a vector of accident probabilities and c is a vector of accident consequences.

HA1 (Monotonicity) If path l_1 is contained in path l_2, then $\zeta^{l_1} \leq \zeta^{l_2}$.

HA2 (Optimality Principle) If $\zeta^{l_2} = \min_{l\in\mathcal{P}_2}\zeta^l$ and path l_1 is contained in path l_2, then $\zeta^{l_1} = \min_{l\in\mathcal{P}_1}\zeta^l$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are the sets of all paths connecting the origin and the destination of path l_1 and path l_2, respectively.

HA3 (Attribute Monotonicity) For any vectors $m \geq 0$ and $n \geq 0$, $\zeta^l(p,c) \leq \zeta^l(p+m, c+n)$ for all path $l \in \mathcal{P}$.

Note that the exact Traditional Risk model (1) violates all three axioms. On the other hand, the approximated Traditional Risk model satisfies all three axioms (Erkut and Verter 1998, Erkut and Ingolfsson 2005). There are many other models that satisfy all three axioms with the approximation. There are some models that satisfy **HA1** and **HA3**, but violate **HA2** (Erkut and Verter 1998). We can show that the CVaR model with approximation as in (5) is one of such models.

PROPOSITION 2. *The CVaR measure with approximation* (5) *satisfies Axioms* ***HA1*** *and* ***HA3*** *for path evaluation models in hazmat routing.*

Figure 1 illustrates an example when CVaR violates **HA2**. For traveling from node 1 to node 4, there are two paths l_1 and l_2, and path l_1 is optimal with respect to CVaR minimization. However, for traveling from node 2 to node 3, path l_3, which is contained in path l_1, is *not* optimal; path l_4 is optimal. Since the CVaR model violates the path-evaluation optimality principle, we cannot use a labeling algorithm to solve the model.

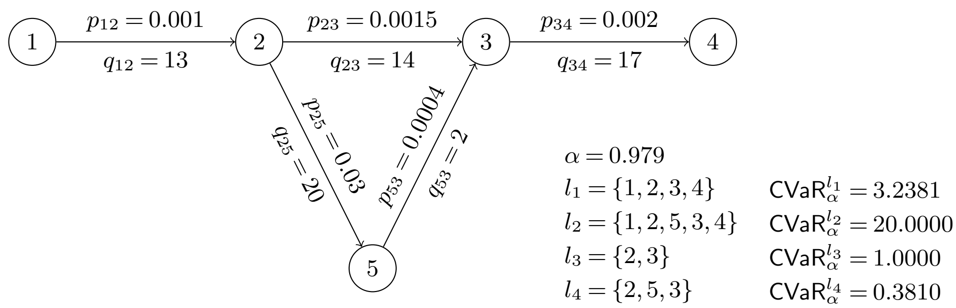


Figure 1 **An example showing that CVaR violates the path-selection optimality principle suggested by Erkut and Verter (1998).**

4. Data Uncertainty and Robust Routing

There are two kinds of uncertain factors in hazmat transportation problems. When we want to minimize the expected accident consequence, we first need to identify the probability distribution of the loss. Such identification is related to this question: In which road segment and at what probability will an accident occur? This is the first uncertain factor.

The second uncertainty comes from the data p_{ij} and c_{ij}, which constitute the distribution of loss. If the data p_{ij} and c_{ij} are provided, we can determine a safe route using various methods. However, for hazmat problems, accurate estimates of the data and their distribution information are rarely available, because hazmat accidents are extremely low-probability events. In addition to this property of hazmat accidents, the accident consequences depend on the weather conditions (Akgun et al. 2007) at the time of the accident, the nature of accidents and hazmat types, and the population migration occurring throughout the day. Moreover, estimation of the environmental impact of an accident can be biased depending on the attitude of the decision maker. Therefore, accident consequences are hard to estimate with accuracy.

Due to the insufficiency of available data, any stochastic-programming-based approach becomes impractical. What we can obtain at best would be interval data of the accident probabilities and consequences. That is, we may be able to obtain the minimum possible and maximum possible values, without distributional information. In such cases, robust optimization methods considering the worst case are most appropriate. For financial portfolio optimization, worst-case CVaR optimization methods have been proposed (Zhu and Fukushima 2009, Čerbáková 2006), but the hazmat routing application has completely different challenges of computability.

To explain the differences, let us denote the uncertain probabilities and consequences by $\tilde{p}_{ij}$ and $\tilde{c}_{ij}$. When interval data are available, we have $\tilde{p}_{ij} \in [p_{ij}, p_{ij} + q_{ij}]$ and $\tilde{c}_{ij} \in [c_{ij}, c_{ij} + d_{ij}]$, where q_{ij} and d_{ij} are positive constants. For more general cases, we can write $\tilde{p} \in \mathcal{U}_p$ and $\tilde{q} \in \mathcal{V}_c$. We call

$\mathcal{U}_p$ and $\mathcal{V}_c$ the set of uncertainty. The uncertain formulation of the Traditional Risk model that minimizes the expected risk is

$$\min_{x\in\Omega} \max_{\tilde{p}\in\mathcal{U}_p, \tilde{c}\in\mathcal{V}_c} \sum_{(i,j)\in\mathcal{A}^l} \tilde{p}_{ij}\tilde{c}_{ij}x_{ij} \tag{8}$$

where

$$\Omega \equiv \left\{ x : \sum_{(i,j)\in\mathcal{A}} x_{ij} - \sum_{(j,i)\in\mathcal{A}} x_{ji} = b_i \quad \forall i \in \mathcal{N}, \text{ and } x_{ij} \in \{0,1\} \quad \forall (i,j) \in \mathcal{A} \right\} \tag{9}$$

and the parameter b_i has the following values:

$$b_i = \begin{cases} 1 & \text{if node } i \text{ is the source} \\ -1 & \text{if node } i \text{ is the sink} \\ 0 & \text{otherwise} \end{cases}$$

Problem (8) a robust shortest-path problem considering the worst-case scenario.

While a few robust shortest-path problems have already been solved (Kouvelis and Yu 1996, Bertsimas and Sim 2003, Chaerani et al. 2005), we face a new class of robust shortest-path problems with two uncertain multiplicative cost coefficients. In the already solved problems, the cost coefficients are considered as a single uncertain-cost vector, such as in $\tilde{c}^T x$. On the other hand, in problem (8), we have two uncertain-cost parameters, $\tilde{p}_{ij}$ and $\tilde{c}_{ij}$, of different characteristics. Data sources for two parameters are different, and the nature of uncertainty is also different; therefore, the two uncertain parameters cannot be regarded as a single-cost parameter in a robust optimization framework. Hence, we need a new method to solve the robust shortest-path problem in (8). It is also unclear if these two parameters are correlated or not (Kwon et al. 2013).

In the next section, we extend the method of Kwon et al. (2013) to solve the worst-case CVaR minimization problems.

5. Worst-case Conditional Value-at-Risk

The worst-case CVaR (WCVaR) minimization can be studied in extension of the robust shortest-path problem (8). When data are uncertain, we define the following WCVaR measure:

$$\begin{aligned} \mathsf{WCVaR}^l_\alpha(\mathcal{U}_p, \mathcal{V}_c) &= \max_{\tilde{p}\in\mathcal{U}_p, \tilde{c}\in\mathcal{V}_c} \min_{r\in\mathbb{R}} \Phi^l_\alpha(r; \tilde{p}, \tilde{c}) \\ &\approx \max_{\tilde{p}\in\mathcal{U}_p, \tilde{c}\in\mathcal{V}_c} \min_{r\in\mathbb{R}} \left\{ r + \left(1 - \sum_{(i,j)\in\mathcal{A}^l} \tilde{p}_{ij}\right)[0-r]^+ + \frac{1}{1-\alpha} \sum_{(i,j)\in\mathcal{A}^l} \tilde{p}_{ij}[\tilde{c}_{ij} - r]^+ \right\} \\ &= \max_{\tilde{p}\in\mathcal{U}_p, \tilde{c}\in\mathcal{V}_c} \min_{r\in\mathbb{R}^+} \left\{ r + \frac{1}{1-\alpha} \sum_{(i,j)\in\mathcal{A}^l} \tilde{p}_{ij}[\tilde{c}_{ij} - r]^+ \right\} \\ &= \min_{r\in\mathbb{R}^+} \max_{\tilde{p}\in\mathcal{U}_p, \tilde{c}\in\mathcal{V}_c} \left(r + \frac{1}{1-\alpha} \sum_{(i,j)\in\mathcal{A}^l} \tilde{p}_{ij}[\tilde{c}_{ij} - r]^+ \right) \end{aligned} \tag{10}$$

where $\tilde{p}$ and $\tilde{c}$ are the uncertain parameters, and $\mathcal{U}$ and $\mathcal{V}$ are compact and convex sets. The switch of 'min' and 'max' in (10) is valid by Theorem 2 of Zhu and Fukushima (2009).

5.1. Axiomatic Study of WCVaR

Since the WCVaR measure in (10) is the worst-case CVaR measure for an uncertain (or ambiguous) probability distribution, the coherence of WCVaR in (10) can be shown directly by the following proposition by Zhu and Fukushima (2009) and Proposition 1:

PROPOSITION 3 **(Zhu and Fukushima 2009)**. *If ρ associated with crisp (or determinate) probability measure P is a coherent risk measure, then the corresponding $\rho_w \equiv \sup_{P\in\mathscr{P}} \rho(X)$ associated with ambiguous probability measure $\mathscr{P}$ remains a coherent risk measure.*

In addition, it is natural that the WCVaR measure satisfies Axioms **HA1** and **HA3**. However, we need to redefine Axiom **HA3** so that it is meaningful with data-uncertainty and WCVaR.

HA3' (Attribute Monotonicity) For any vectors $m \geq 0$ and $n \geq 0$, $\mathsf{WCVaR}^l_\alpha(\mathcal{U}_p, \mathcal{V}_c) \leq \mathsf{WCVaR}^l_\alpha(\mathcal{U}_p + m, \mathcal{V}_c + n)$ for all paths $l \in \mathcal{P}$ and all $\alpha \in (0,1)$, where

$$\mathcal{U}_p + m = \{p + m : p \in \mathcal{U}_p\}$$
$$\mathcal{V}_c + n = \{c + n : c \in \mathcal{V}_c\}$$

We can also consider the following alternative:

HA3" (Attribute Monotonicity) For any compact and convex sets $\mathcal{U}_p$, $\mathcal{V}_c$, $\mathcal{U}'_p$ and $\mathcal{V}'_c$, $\mathsf{WCVaR}^l_\alpha(\mathcal{U}_p, \mathcal{V}_c) \leq \mathsf{WCVaR}^l_\alpha(\mathcal{U}'_p, \mathcal{V}'_c)$ for all path $l \in \mathcal{P}$ and all $\alpha \in (0,1)$, where $\mathcal{U}_p \preccurlyeq \mathcal{U}'_p$ and $\mathcal{V}_c \preccurlyeq \mathcal{V}'_c$ meaning that

$$p \leq p' \quad \forall p \in \mathcal{U}_p,\ p' \in \mathcal{U}'_p \quad (\text{or, } \sup\ \mathcal{U}_p \leq \inf\ \mathcal{U}'_p)$$
$$c \leq c' \quad \forall c \in \mathcal{V}_c,\ c' \in \mathcal{V}'_c \quad (\text{or, } \sup\ \mathcal{V}_c \leq \inf\ \mathcal{V}'_c)$$

With these axioms, we present the following result:

PROPOSITION 4. *When $\mathcal{U}$ and $\mathcal{V}$ are compact and convex sets, the worst-case CVaR (WCVaR) measure*

$$\mathsf{WCVaR}^l_\alpha(\mathcal{U}_p, \mathcal{V}_c) = \min_{r\in\mathbb{R}^+} \max_{\tilde{p}\in\mathcal{U}_p, \tilde{c}\in\mathcal{V}_c} \left\{ r + \frac{1}{1-\alpha} \sum_{(i,j)\in\mathcal{A}^l} \tilde{p}_{ij}[\tilde{c}_{ij} - r]^+ \right\}$$

*is a coherent risk measure and satisfies Axioms **HA1** and **HA3** (both **HA3'** and **HA3"**).*

The WCVaR minimization problem applied to hazmat transportation is as follows:

$$\min_{l\in\mathcal{P}} \mathsf{WCVaR}^l_\alpha(\mathcal{U}_p, \mathcal{V}_c) = \min_{l\in\mathcal{P}} \min_{r\in\mathbb{R}^+} \max_{\tilde{p}\in\mathcal{U}_p, \tilde{c}\in\mathcal{V}_c} \left(r + \frac{1}{1-\alpha} \sum_{(i,j)\in\mathcal{A}^l} \tilde{p}_{ij}[\tilde{c}_{ij} - r]^+ \right) \tag{11}$$

$$= \min_{r\in\mathbb{R}^+} \left(r + \frac{1}{1-\alpha} \min_{x\in\Omega} \max_{\tilde{p}\in\mathcal{U}_p, \tilde{c}\in\mathcal{V}_c} \sum_{(i,j)\in\mathcal{A}} \tilde{p}_{ij}[\tilde{c}_{ij} - r]^+ x_{ij} \right) \tag{12}$$

where the set Ω is defined in (9). For each r, the WCVaR minimization problem requires us to solve a robust shortest-path problem with two uncertain multiplicative cost coefficients. We will extend the solution approach proposed by Kwon et al. (2013) for solving the sub robust shortest-path problem.

5.2. Box-Constrained Uncertainty Set

Let us consider the uncertain parameters $\tilde{p}$ and $\tilde{c}$ given in the following budgeted box-constrained uncertainty set, as considered by Kwon et al. (2013).

$$\tilde{p}_{ij} = p_{ij} + q_{ij}u_{ij} \tag{13}$$

$$\tilde{c}_{ij} = c_{ij} + d_{ij}v_{ij} \tag{14}$$

where q_{ij} and d_{ij} are given constants,

$$u_{ij} \in \mathcal{U} = \left\{ u : 0 \le u_{ij} \le 1 \quad \forall (i,j), \quad \sum_{(i,j)\in\mathcal{A}} u_{ij} \le \Gamma_u \right\} \tag{15}$$

$$v_{ij} \in \mathcal{V} = \left\{ v : 0 \le v_{ij} \le 1 \quad \forall (i,j), \quad \sum_{(i,j)\in\mathcal{A}} v_{ij} \le \Gamma_v \right\} \tag{16}$$

and Γ_u and Γ_v are positive integers. An intuitive description of these parameter is given in Section 6.1. The parameters Γ_u and Γ_v are called the *budgets of uncertainty*, and represent the level of ambiguity in the data. Increasing the values of Γ_u and Γ_v increases the level of robustness in the objective (Bertsimas and Sim 2003). Replacing the uncertain parameters $\tilde{p}$ and $\tilde{c}$ in (11) with the corresponding budgeted box-constrained set, we obtain:

$$\min_{r\in\mathbb{R}^+} \left(r + \frac{1}{1-\alpha} \min_{x\in\Omega} \max_{u\in\mathcal{U}, v\in\mathcal{V}} \sum_{(i,j)\in\mathcal{A}} (p_{ij} + q_{ij}u_{ij})[(c_{ij} + d_{ij}v_{ij}) - r]^+ x_{ij} \right) \tag{17}$$

We first observe that v at optimum is binary.

Lemma 1. *There always exists a binary v that is a solution to the WCVaR minimization problem* (17)*. In particular, for any given r, x, and u, the maximization problem of v*

$$\max_{v\in\mathcal{V}} \sum_{(i,j)\in\mathcal{A}} (p_{ij} + q_{ij}u_{ij})[(c_{ij} + d_{ij}v_{ij}) - r]^+ x_{ij} \tag{18}$$

has a binary solution.

Note that $[(c_{ij} + d_{ij}v_{ij}) - r]^+$ takes values in the interval $[[c_{ij} - r]^+, [c_{ij} + d_{ij} - r]^+]$. Let us define

$$e_{ij}(r) = [c_{ij} - r]^+ \tag{19}$$

$$f_{ij}(r) = [c_{ij} + d_{ij} - r]^+ - [c_{ij} - r]^+ \tag{20}$$

A linearizing transformation from $[(c_{ij} + d_{ij}v_{ij}) - r]^+$ to $e_{ij}(r) + f_{ij}(r)v_{ij}$ changes the shape of the objective function; the former is nonlinear and the latter is linear in v_{ij}. However, when they are maximized with respect to v over the set $\mathcal{V}$, they have same binary solutions by Lemma 1. Therefore, we can rewrite the inner min-max problem in (17) equivalently as follows:

$$\begin{aligned}&\min_{x\in\Omega}\ \max_{u\in\mathcal{U},v\in\mathcal{V}}\sum_{(i,j)\in\mathcal{A}}(p_{ij}+q_{ij}u_{ij})[(c_{ij}+d_{ij}v_{ij})-r]^{+}x_{ij}\\ =&\min_{x\in\Omega}\ \max_{u\in\mathcal{U},v\in\mathcal{V}}\sum_{(i,j)\in\mathcal{A}}(p_{ij}+q_{ij}u_{ij})(e_{ij}(r)+f_{ij}(r)v_{ij})x_{ij}\equiv\mathsf{RSP}(r)\end{aligned}\tag{21}$$

We observe that the inner maximization problem in (21) is a non-convex disjoint bilinear program for any given x and r, for which an optimal solution exists at an extreme point (Floudas and Visweswaran 1994). Therefore we can obtain binary optimal solutions u and v. We obtain the following result.

Proposition 5. *There exists an optimal solution, r^*, to the WCVaR minimization problem* (17) *in the following set:*

$$r^*\in\mathcal{R}\equiv\{0\}\cup\{c_{ij}:(i,j)\in\mathcal{A}\}\cup\{c_{ij}+d_{ij}:(i,j)\in\mathcal{A}\}\tag{22}$$

Proposition 5 indicates that for each value of r in the set $\mathcal{R}$, we need to solve a corresponding robust shortest-path problem $\mathsf{RSP}(r)$ to obtain an optimal solution to the WCVaR minimization problem.

For any fixed r, $\mathsf{RSP}(r)$ is an instance of the robust shortest path problems with two multiplicative uncertain coefficients considered by Kwon et al. (2013). Each $\mathsf{RSP}(r)$ can be reformulated as a mixed integer linear programming problem that can be solved by optimization solvers like CPLEX or can be solved by a finite number of (deterministic) shortest path problems using methods proposed by Kwon et al. (2013).

5.3. Computational Scheme for WCVaR Minimization

Proposition 5 enables us to write the WCVaR minimization problem (17) as follows:

$$\mathsf{WCVaR}^*_\alpha=\min_{r\in\mathcal{R}}\mathsf{RSP}(r)$$

where $\mathcal{R}$ is defined in (22). In an actual implementation of the solution procedure for WCVaR, we can reduce the number of r values to be considered. Since $\mathsf{RSP}(r)$ always has nonnegative value, it is unnecessary to consider r values greater than the minimum value found so far; therefore it is beneficial to examine the set $\mathcal{R}$ in ascending order. Optimality of the following scheme is guaranteed by Proposition 5:

A Computational Scheme for the WCVaR minimization problem (17)

Step 0. Let $\mathcal{R}$ be an ascendingly ordered set of $\{0\}\cup\{c_{ij}:(i,j)\in\mathcal{A}\}\cup\{c_{ij}+d_{ij}:(i,j)\in\mathcal{A}\}$ and r_k denote k-th smallest element of $\mathcal{R}$ where $\hat{r}_0=0$. Set $w^\sharp\leftarrow\infty$ and $k\leftarrow 0$.

Step 1. If $r_k \geq w^\sharp$, terminate the algorithm.

Step 2. For r_k, solve a robust shortest path problem $\mathsf{RSP}(r_k)$ as defined in (21) using a method of Kwon et al. (2013). Then compute

$$w^k = r_k + \frac{1}{1-\alpha}\,\mathsf{RSP}(r_k)$$

If $w^k < w^\sharp$, set $w^\sharp \leftarrow w^k$.

Step 3. Update $k \leftarrow k+1$ and repeat Steps 1 and 2, until $k = 2|\mathcal{A}|+1$.

At termination, we find the WCVaR value as $w^\sharp$, and the optimal path as the path obtained by the corresponding robust shortest-path problem.

5.4. Properties of WCVaR Minimization

Depending on the value of α, the WCVaR model can be identical to the worse-case version of one of classic models in Table 2; therefore, the WCVaR model is more general than other models.

Proposition 6. *For sufficiently small $\alpha > 0$, the WCVaR minimization is equivalent to the worst-case Traditional Risk (TR) model. That is*

$$\min_{l\in\mathcal{P}} \mathsf{WCVaR}^l_\alpha = \min_{l\in\mathcal{P}} \max_{\tilde{p}\in\mathcal{U}_p, \tilde{c}\in\mathcal{V}_c} \sum_{(i,j)\in\mathcal{A}^l} \tilde{p}_{ij}\tilde{c}_{ij}$$

Please note that in hazmat applications sufficiently small α as in Proposition 6 can be as large as 0.999933. We will observe this later with a case study (Table 3).

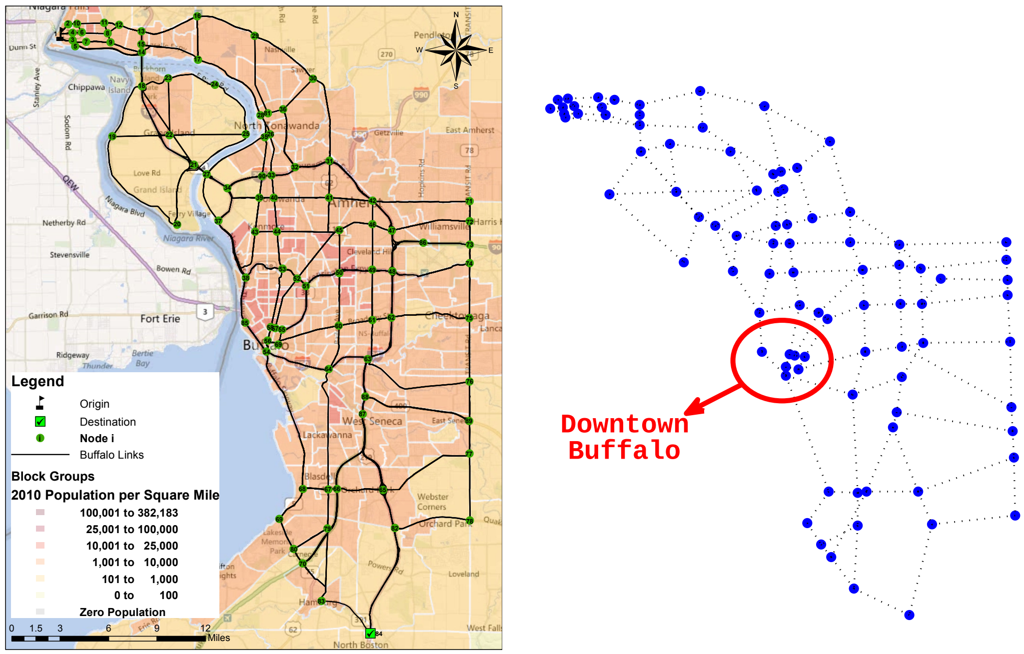
Proposition 7. *For sufficiently large $\alpha < 1$, the WCVaR minimization model is equivalent to the worst-case Maximum Risk (MM) model. That is*

$$\min_{l\in\mathcal{P}} \mathsf{WCVaR}^l_\alpha = \min_{l\in\mathcal{P}} \max_{\tilde{c}\in\mathcal{V}_c} \max_{(i,j)\in\mathcal{A}^l} \tilde{c}_{ij} = \min_{l\in\mathcal{P}} \max_{(i,j)\in\mathcal{A}^l} (c_{ij}+d_{ij})$$

While α represents the probability threshold for WCVaR, the above properties indicate that one can examine various α values to generate alternative paths for hazmat routing. While a proper choice of α can be tricky to practitioners, any “wrong” choice of α still provides a valid result based on one of classic models; e.g., the worst-case TR (WTR) model when α was mistakenly chosen small as 0.999933.

6. Numerical Experiments

Through numerical experiments in a realistic road network, we demonstrate the characteristics of least-WCVaR paths and confirm the analytical findings.



(a) Actual Network with Population Densities

(b) Simplified Representation

Figure 2 Alternative Representations of the Case Study Network

6.1. Test Network and Data Analysis

The proposed model was tested in a portion of an actual vehicular road network in Buffalo, New York, USA, as shown in Figure 2. The Buffalo network consists of 90 nodes and 149 arcs, a unique origin-destination pair (OD pair), and a single hazmat shipment that needs to be transported from the origin to the destination. The population data were obtained from the U.S. Census Bureau (US Census Bureau 2010).

For every arc we need to specify two attributes: accident probabilities (p_{ij}) and accident consequences (c_{ij}). To obtain the nominal accident probabilities, we used the following formula:

$$p_{ij} = 3.16622 \times 10^{-7} \times (\text{length of arc } (i,j)) \tag{23}$$

where 3.16622×10^{-7} is the hazmat accident rate per mile/vehicle (Federal Motor Carrier Safety Administration 2001). Hence, p_{ij} can be interpreted as the expected accident probability on arc (i,j).

For the computation of the accident consequences c_{ij}, we considered the population density in the hazmat impact zone. Specifically, we estimated the population density in a circle of radius λ, which is equal to the hazmat spread radius as shown in Figure 3, that is commonly considered in the literature (Erkut and Verter 1998, Erkut et al. 2007). The formula used for the computation of the accident consequences was the following:

$$c_{ij} = \pi \cdot \lambda^2 \cdot \rho_{ij} \tag{24}$$

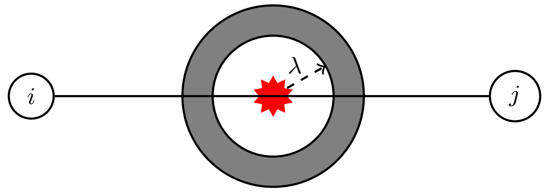


Figure 3 **Hazmat accident endangered area described by a circle of radius λ in arc (i,j); the shaded area is the uncertainty band.**

where ρ_{ij} is the average population density along arc (i,j) and λ is assumed to be equal to 1 mile. The value of λ was selected based on the recommendations of the Emergency Response Guidebook (2012) for the length of the evacuation radius in the case of an accident involving hazmat, which ranges between 0.5 and 1 mile depending on the type of the hazmat.

Although we used the population exposure in (24) as it has the top priority in practice (Federal Motor Carrier Safety Administration 2007), c_{ij} can include any measure of accident consequences such as effects on commerce, delays in transportation, and damages to the environment (Shaver and Kaiser 1998). However measuring such factors requires multi-dimensional considerations of population density, road type, type/quantity of hazmat, emergency response capability, terrain, climatic conditions, etc. (Shaver and Kaiser 1998), and sufficient information may be unavailable to decision makers. In such a case, WCVaR has an advantage over other routing methods in Table 2 that generate a single route. When the WCVaR model with the population exposure as c_{ij} generates a route that is unacceptable for environmental reasons, the decision maker can adjust the probability threshold α to generate alternate routes. Such flexibility is unavailable for many other routing models. We will demonstrate this point through numerical examples later in this section.

We specified the values for the worst-case deviations of accident probability and accident consequence: q_{ij} and d_{ij}, respectively. The following worst-case scenarios were assumed:

1. Accident probabilities can at most be doubled, i.e. $q_{ij} = p_{ij}$.
2. Hazmat spread radius can be as far as 1.5 miles. That is, an increase of 0.5 mile in the radius λ of the endangered area as shown in the shaded area in Figure 3; therefore, $d_{ij} = \pi \cdot (1.5)^2 \cdot \rho_{ij} - c_{ij}$.

Additionally, we set the values for Γ_u and Γ_v, which represent the ambiguity level of data or the budget of uncertainty, as $\Gamma_u = 8$ and $\Gamma_v = 5$. We present a sensitivity study of these parameters in Section 6.3.

All computations were performed in Matlab 2012a on a 2.8 GHz Intel Core 2 Duo computer system with 8 GB memory.

6.2. Comparison of Models

With the above data on hand, the WCVaR model resulted in 14 optimal paths for confidence levels in the interval $[0,1)$. The model was tested for 101 different confidence level values and the

Table 3 Optimal paths - WCVaR model using $q_{ij} = p_{ij}$, $d_{ij} = \pi(1.5)^2 \cdot \rho_{ij} - c_{ij}$, $\Gamma_u = 8$ and $\Gamma_v = 5$

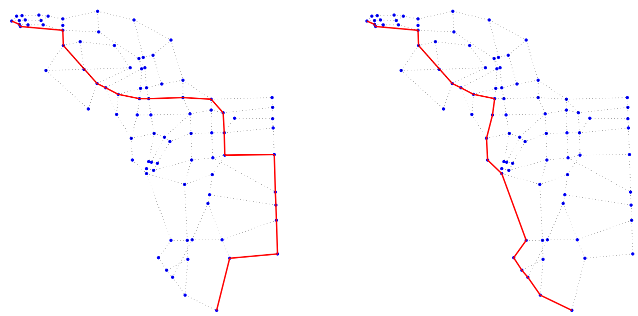
Confidence Level α	Optimal WCVaR Route
$[0, 0.999933)$	1,3,5,14,18,21,27,34,39,40,41,42,47,48,62,75,76,89,77,78,82,84
$[0.999933, 0.999944)$	1,3,5,14,18,19,22,21,27,34,39,40,41,42,47,48,62,75,76,89,77,78,82,84
$[0.999944, 0.999955)$	1,3,5,14,18,21,27,34,39,40,41,42,47,48,62,75,76,89,77,78,82,84
$[0.999955, 0.999965)$	1,3,5,14,18,19,22,21,27,34,39,40,41,42,47,48,62,75,76,89,77,78,82,84
$[0.999965, 0.999968)$	1,3,5,14,18,21,27,34,39,40,41,42,47,48,62,75,76,89,77,78,82,84
$[0.999968, 0.999972)$	1,3,5,14,18,21,27,34,39,40,41,42,47,72,73,74,48,62,75,76,89,77,78,82,84
$[0.999972, 0.999973)$	1,3,5,14,18,23,24,25,21,27,34,39,40,41,42,71,72,73,74,48,62,75,76,89,77,65,82,84
$[0.999973, 0.999977)$	1,3,5,14,18,23,24,25,21,27,34,39,40,41,42,71,72,73,74,48,62,75,76,89,77,78,82,84
$[0.999977, 0.999986)$	1,3,5,14,18,22,21,27,34,39,40,41,42,71,72,73,74,48,62,75,76,89,77,78,82,84
$[0.999986, 0.999990)$	1,3,5,14,18,23,24,25,22,21,27,34,39,40,41,42,71,72,73,74,48,62,75,76,89,77,78,82,84
$[0.999990, 0.999992)$	1,3,5,14,18,23,22,21,27,34,39,40,41,42,71,72,73,74,75,76,89,77,65,66,68,79,83,84
$[0.999992, 0.999993)$	1,3,5,14,18,23,24,25,21,27,34,39,43,38,85,54,67,69,80,70,83,84
$[0.999993, 0.999997)$	1,4,3,5,14,17,28,35,27,34,39,43,38,85,54,67,69,80,70,83,84
$[0.999997, 1)$	1,3,5,14,18,21,27,34,39,43,38,85,54,67,69,80,70,83,84

maximum computational time for a single α-value was less than 21 seconds. The least-WCVaR paths are presented in Table 3 and sample paths graphically in Figures 4 and 5.

An interesting observation from Table 3 is that the least-WCVaR path for $\alpha \geq 0.999997$ that is identical to the least- worst-case MM (WMM) path, as shown in Figure 4b, is passing through the highly populated area of downtown Buffalo. This is a rather unexpected result, since for confidence levels close to 1 the level of robustness is increasing; therefore, one would expect that the proposed path would avoid high populated areas for high α-values. This behavior is due to the structure of the Buffalo network. Note that, the path has its one side next to Niagara River. Hence, the area affected by a potential accident along that path is cut in half, because we considered population exposure only in our current computation of accident consequence. This result encourages us to consider more flexible routing methods than simple methods like the MM or WMM models. Although the downtown area faces the waterway beyond the environmental impact, it would still be unsafe to transport hazmat through the downtown area. Therefore, a decision maker would need alternatives, which the WCVaR (and CVaR) model can provide.

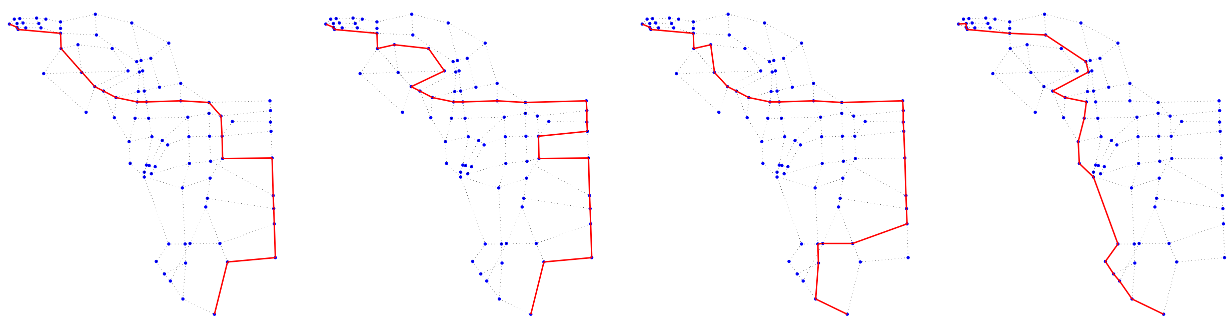
Figure 4 demonstrates the proposed paths from the WTR and WMM models for comparison purposes. As stated in Proposition 6, WCVaR for small values of α is equivalent with the WTR model. Figure 4a illustrates that the paths from the two models are indeed same. Similarly, the WCVaR path for large values of confidence level—specifically for $\alpha \geq 0.999997$—is equivalent to the WMM path (proposing the same path, as shown in Figure 4b), verifying Proposition 7.

From the results presented in Figures 4 and 5, the flexibility of the WCVaR model is apparent. When the WTR or WMM models are used, the decision makers have a unique path in hand with no other alternatives to choose from, constraining their flexibility. On the contrary, using the WCVaR model, the decision maker has the ability, by altering the confidence level value, to obtain different paths covering various levels of risk-attitudes as demonstrated in Figure 5.



(a) WTR path, WCVaR path for $\alpha = 0$ (b) WMM path, WCVaR path for $\alpha = 0.999997$

Figure 4 **Optimal Paths from the Worst-case TR (WTR) and Worst-case MM (WMM) models and their relationship with the WCVaR model for "small" α and "big" α respectively, using $q_{ij} = p_{ij}$, $d_{ij} = \pi(1.5)^2 \cdot \rho_{ij} - c_{ij}$, $\Gamma_u = 8$ and $\Gamma_v = 5$**



(a) $\alpha = 0.999965$ (b) $\alpha = 0.999973$ (c) $\alpha = 0.999990$ (d) $\alpha = 0.999993$

Figure 5 **Samples of Least-WCVaR Paths using $q_{ij} = p_{ij}$, $d_{ij} = \pi(1.5)^2 \rho_{ij} - c_{ij}$, $\Gamma_u = 8$ and $\Gamma_v = 5$**

Table 4 presents a more thorough comparison among VaR, CVaR and WCVaR models with respect to various risk measure values and the length of paths. We also observe in Table 4 that, at $\alpha = 0.999995$, the VaR path produces relatively higher values in most risk measures, compared to other α values. This result indicates that the VaR model may produce an undesirable path by ignoring long tail. On the contrary, the behavior of the CVaR path at the same α value shows smooth changes. We can also confirm this point by comparing the VaR and CVaR models in terms of the CVaR risk measure. As shown in Figure 6a, the performance gap was as large as 17.6%.

We are also interested in how much value the WCVaR model would add over the VaR/CVaR models. If the CVaR model would already produce good paths in terms of the WCVaR risk measure, then it would weaken our motivation to consider the data uncertainty and tolerate computational complexity of the WCVaR model. Based on the data presented in Table 4, we provided a chart in Figure 6b. When the VaR and WCVaR paths are compared in terms of the WCVaR risk measure, the WCVaR paths showed significant improvements up to 34.5%. When compared with the CVaR paths, the WCVaR paths were better off by up to 17.4%. These observations signal that data uncertainty should not be overlooked.

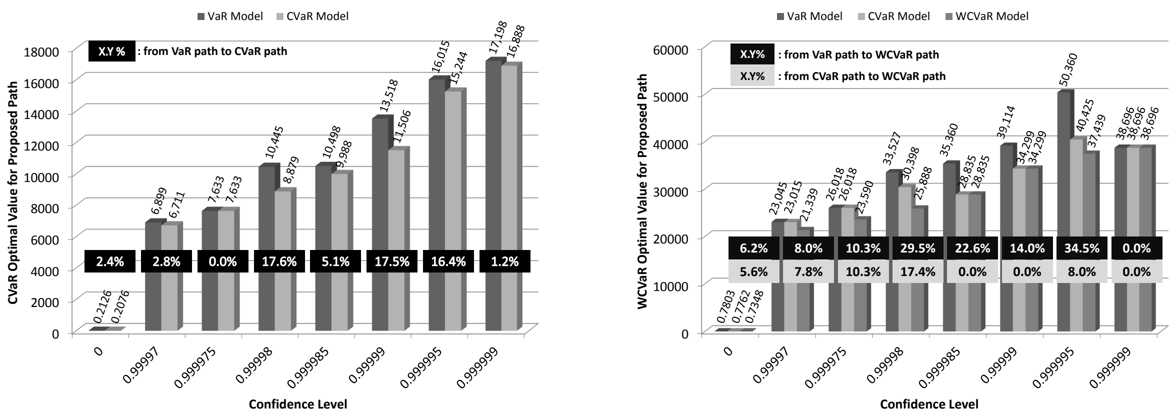
Table 4 Comparison of VaR, CVaR and WCVaR models, when $q_{ij} = p_{ij}$, $d_{ij} = \pi(1.5)^2 \cdot \rho_{ij} - c_{ij}$, $\Gamma_u = 8$ and $\Gamma_v = 5$

Model	Confidence Level α	Risk Measure Values of each given path							Path Properties	
		TR	MM	WTR	WMM	VaR$_\alpha$	CVaR$_\alpha$	WCVaR$_\alpha$	Number of arcs	Distance (miles)
VaR	0	0.2126	18033	0.7803	40573	0	0.2126	0.7803	17	35.37
	0.999970	0.2130	18033	0.7798	40573	1093	6899	23045	19	38.05
	0.999975	0.2076	18033	0.7762	40573	2426	7633	26018	16	35.70
	0.999980	0.2662	18845	0.8441	42400	3229	10445	33527	22	47.20
	0.999985	0.2076	18033	0.7762	40573	4023	10498	35360	16	35.70
	0.999990	0.2076	18033	0.7762	40573	4679	13518	39114	16	35.70
	0.999995	0.3279	27663	0.9243	62241	10680	17755	50360	23	54.33
	0.999999	0.2380	17198	0.8116	38696	17198	17198	38696	18	38.00
CVaR	0	0.2076	18033	0.7762	40573	0	0.2076	0.7762	16	35.70
	0.999970	0.2076	18033	0.7762	40573	1722	6711	23015	16	35.70
	0.999975	0.2081	18033	0.7757	40573	2426	7633	26018	18	38.38
	0.999980	0.2113	18033	0.7803	40573	4023	8879	30398	17	35.80
	0.999985	0.2739	18845	0.8190	42400	6213	9988	28835	25	54.20
	0.999990	0.2683	18845	0.8094	42400	7575	11506	34299	25	54.00
	0.999995	0.2536	18845	0.7979	42400	10889	15244	40425	22	47.10
	0.999999	0.2380	17198	0.8116	38696	15646	16888	38696	18	38.00
WCVaR	0	0.2399	18845	0.7348	42400	0	0.2399	0.7348	21	46.70
	0.999970	0.2718	18845	0.8021	42400	3122	7528	21339	24	53.80
	0.999975	0.2831	18845	0.8242	42400	4056	8073	23590	27	60.20
	0.999980	0.2683	18845	0.8094	42400	6102	9025	25888	25	54.00
	0.999985	0.2683	18845	0.8094	42400	6213	9988	28835	25	54.00
	0.999990	0.3112	18845	0.8952	42400	7575	12376	34299	27	51.70
	0.999995	0.3375	17198	1.0215	38696	15157	16016	37439	20	42.98
	0.999999	0.2466	17198	0.8202	38696	15646	16474	38696	18	38.00

Moreover, Table 4 provides us with information on the length of the paths proposed by each model. When WCVaR paths are compared with VaR and CVaR paths at the respective confidence levels, the WCVaR paths are usually longer especially at lower confidence levels. We may interpret this observation as the robust nature of the WCVaR model induces more circuitous routes. When the CVaR path is already risk-averse with larger α value, the consideration of data uncertainty impacts less.

6.3. Sensitivity Analysis

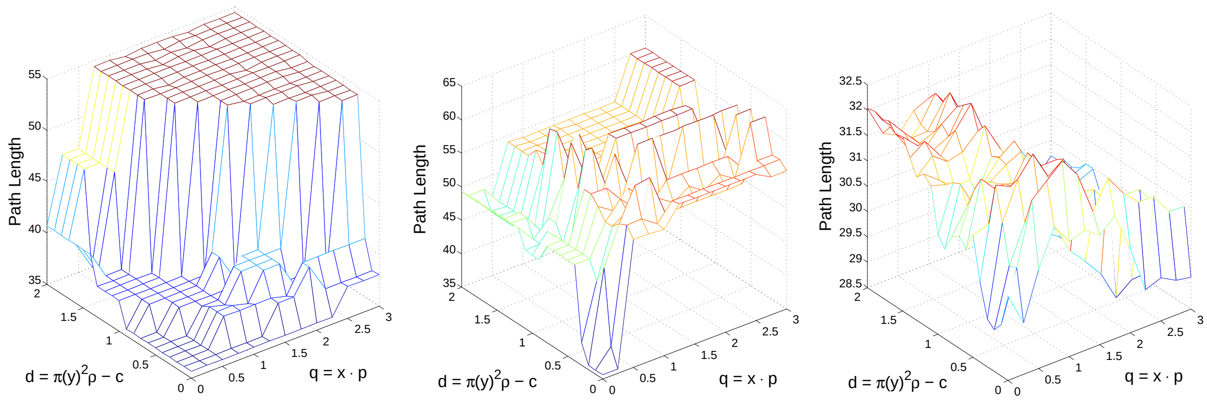
Risk-averse routing in hazmat transportation usually results in a very circuitous path, because one can usually avoid high accident consequences by detouring them. In this section, we discuss the relationship between the length of the least-WCVaR paths and various model parameters that are relevant to risk-averseness. In the WCVaR model, there are several parameters: α, q_{ij}, p_{ij}, d_{ij}, c_{ij}, Γ_u and Γ_v. Note that p_{ij} and c_{ij} are obtained from the Buffalo network, therefore, the decision maker has no control over them. On the other hand, we can control the level of robustness by altering the values of α, q_{ij}, d_{ij}, Γ_u and Γ_v. Keeping this fact in mind, sensitivity analyses were conducted keeping p_{ij} and c_{ij} fixed, and changing the values of α, q_{ij}, d_{ij}, Γ_u and Γ_v.



(a) Improvements from the least-VaR to the least-CVaR path, measured in CVaR_α

(b) Improvements from the least-VaR and least-CVaR paths to the least-WCVaR path, measured in WCVaR_α

Figure 6 Improvement in CVaR and WCVaR risk measures

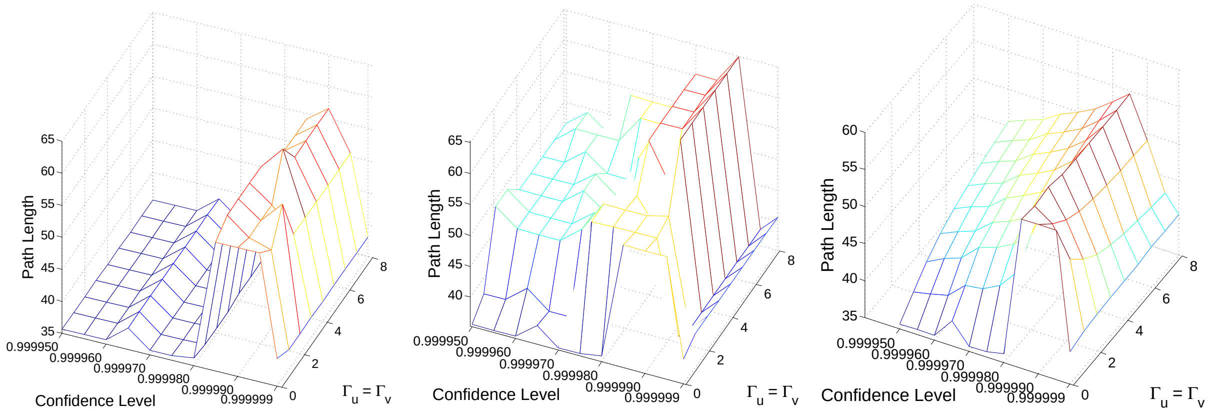


(a) $\alpha = 0.999970$, $\Gamma_u = 1$, $\Gamma_v = 1$ (b) $\alpha = 0.999975$, $\Gamma_u = 5$, $\Gamma_v = 8$ (c) Average Path Length

Figure 7 Sensitivity Analyses for the WCVaR Model (q vs. d vs. Path Length)

Since transportation cost is an important factor for hazmat carriers, it is important to understand how each parameter impacts the length of the resulting path. We tested the WCVaR model using 40 combinations of α, Γ_u and Γ_v and for each combination q_{ij} was in the interval $[0, 3p_{ij}]$ with incremental step size $0.25p_{ij}$, and d_{ij} was in the interval $[0, \pi(3)^2\rho_{ij} - c_{ij}]$ with incremental size 0.1 miles for the hazmat spread radius. For each combination of the parameters' values we measured the length of the proposed path. In other words, we measure how circuitous the model becomes at various levels of robustness. A sample of the results is presented in Figure 7.

First, we observe that q_{ij} and d_{ij} show no unilateral relationship with the path length. At certain α, Γ_u and Γ_v settings, there could be a positive trend (Figure 7a for $\alpha = 0.999970, \Gamma_u = \Gamma_v = 1$), and at some other settings, there could be no trend (Figure 7b for $\alpha = 0.999975, \Gamma_u = 5, \Gamma_v = 8$). However, looking at the scale of the path-length-axis, we can observe that there are significant jumps from Figure 7a to Figure 7b. Motivated by this observation, we generated Figure 7c to see if



(a) $q=0$, $d=\pi(1.10)^2\rho-c$ (b) $q=p$, $d=\pi(2)^2\rho-c$ (c) Average Path Length

Figure 8 Sensitivity Analyses for the WCVaR Model (α vs. $\Gamma_u=\Gamma_v$ vs. Path Length)

there exist a trend on the average path length for each pair of q and d. As shown by Figure 7c no trend can be identified on the average path length.

Figure 8 demonstrates the relationship between the budgets of uncertainty Γ_u and Γ_v, the confidence level α and the travel distance of the proposed path. These plots were obtained by setting Γ_u equal to Γ_v in the interval $[0,8]$ and varying α values from 0.999950 to 0.999999. The vectors q_{ij} and d_{ij} were altered as described previously. As shown in Figures 8a and 8b, there exists a positive trend between the uncertainty budgets and confidence level with the path length. That is up to the point where the confidence level is approaching 1. At that point, due to the special structure of our case-study network, the proposed path length is shorter than before. Figure 8c presents the relation between Γ_u and Γ_v, the confidence level α and the average travel distance of the proposed path for each combination. From the latter figure we can generalize our observation that the path becomes more circuitous with larger Γ_u $(=\Gamma_v)$ values. This is also true with larger α; however, when α is very large and approaching 1, the least-WCVaR path becomes short. This is because the Buffalo network has waterfronts and only one arc becomes critical when α is very large.

7. Conclusions

In this paper, we provided an axiomatic study of conditional value-at-risk (CVaR), a concept first presented in Toumazis et al. (2013), and we formally introduced the concept of worst-case CVaR (WCVaR) for determining safe transport routes for hazardous materials (hazmat). We emphasize that the WCVaR model inherits the analytical and computational characteristics of the CVaR model. WCVaR is a coherent risk measure for use as an appropriate hazmat routing criterion. In addition, WCVaR offers a flexible, risk-averse, and computationally tractable routing method. While CVaR and WCVaR both violate the path selection optimality principle of Erkut and Verter (1998), we devised a tractable computational method to determine least-WCVaR paths. We confirmed our analytical findings via numerical experiments in a realistic road network in Buffalo, New York.

In general, the more risk-averse the hazmat carrier, the more circuitous will be the path chosen. This is natural, since it is usually possible to avoid highly populated areas by detouring city centers. Our experience mostly agrees with this consensus. However, in extreme cases, as when the maximum risk model is used, or when CVaR/WCVaR models are used with a very large confidence level, the consensus may fail to be true. In the Buffalo network that we considered in this paper, those extreme risk models provided shorter routes that passed through the downtown central business area. This was mainly because we did not consider the environmental consequences of an accident and the Buffalo downtown area faces the Niagara River waterways, which reduces by half the accident consequences in the area. However, driving through the downtown area still would be an invalid choice for hazmat carriers. In fact, this observation makes the flexibility of the CVaR/WCVaR models more attractive, since a decision maker can easily generate alternatives with the CVaR/WCVaR models.

The scope of our current work is to address routing problems in a static network. However, a consideration of the changes in population density throughout the day will improve the realism and applicability of the model. We plan to develop a dynamic WCVaR model similar to that developed for the CVaR model (Toumazis and Kwon 2013) as a future extension to this research. Such a dynamic WCVaR model network would allow the incorporation of population migration throughout the day and its effect on accident probabilities and consequences.

The main limitation of the CVaR/WCVaR models considered in this paper is that they address the problem of routing a single hazmat shipment with a unique origin-destination (OD) pair. Even though this problem can efficiently address local route planning problems for each shipment individually, network regulators will be interested in minimizing overall risk exposure with multiple shipments and multiple OD pairs. Addressing the hazmat routing problem from a network regulators perspective using CVaR/WCVaR models is a potential future work.

Another possible consideration for the future is different types of uncertainty sets. While we considered box-constrained uncertainty sets in this paper, consideration might be given, for example, to ellipsoidal uncertainty sets. We know that the robust shortest-path problem itself becomes NP-hard with such sets, therefore an approximate or heuristic method must be developed to solve the corresponding WCVaR minimization problem. The challenge of the two uncertain multiplicative cost coefficients should also be addressed.

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Appendix. Proofs of Propositions and Lemmas

Proof of Proposition 1: We provide a proof for each axiom of **CA1**, **CA2'**, and **CA3**.

(CA1) Letting $r = s - m$, we obtain the desired result.

(CA2') Let

$$s^* = \arg\min_{s\in\mathbb{R}^+}\left[s + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_1}} p_{ij}[c_{ij}-s]^+\right]$$
$$t^* = \arg\min_{t\in\mathbb{R}^+}\left[t + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_2}} p_{ij}[c_{ij}-t]^+\right]$$

Then

$$\begin{aligned}
&\min_{r\in\mathbb{R}^+}\left[r + \frac{1}{1-\alpha}\left\{\sum_{(i,j)\in\mathcal{A}^{l_1}} p_{ij}[c_{ij}-r]^+ + \sum_{(i,j)\in\mathcal{A}^{l_2}} p_{ij}[c_{ij}-r]^+\right\}\right]\\
&\le s^* + t^* + \frac{1}{1-\alpha}\left\{\sum_{(i,j)\in\mathcal{A}^{l_1}} p_{ij}[c_{ij}-s^*-t^*]^+ + \sum_{(i,j)\in\mathcal{A}^{l_2}} p_{ij}[c_{ij}-s^*-t^*]^+\right\}\\
&\le s^* + t^* + \frac{1}{1-\alpha}\left\{\sum_{(i,j)\in\mathcal{A}^{l_1}} p_{ij}[c_{ij}-s^*]^+ + \sum_{(i,j)\in\mathcal{A}^{l_2}} p_{ij}[c_{ij}-t^*]^+\right\}\\
&= \min_{s\in\mathbb{R}^+}\left[s + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_1}} p_{ij}[c_{ij}-s]^+\right] + \min_{t\in\mathbb{R}^+}\left[t + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_2}} p_{ij}[c_{ij}-t]^+\right]
\end{aligned}$$

(CA3) Letting $\lambda r = s$, we can show that

$$\min_{s\in\mathbb{R}^+}\left[s + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l}} p_{ij}[\lambda c_{ij}-s]^+\right] = \lambda\min_{r\in\mathbb{R}^+}\left[r + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l}} p_{ij}[c_{ij}-r]^+\right]$$

for all $\lambda \ge 0$.

$\square$

Proof of Proposition 2: We provide a proof for each axiom.

(HA1) We have

$$\begin{aligned}
\mathsf{CVaR}_\alpha^{l_2} &= \min_{s\in\mathbb{R}^+}\left[s + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_2}} p_{ij}[c_{ij}-s]^+\right]\\
&= \min_{s\in\mathbb{R}^+}\left[s + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_1}} p_{ij}[c_{ij}-s]^+ + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_2}\setminus\mathcal{A}^{l_1}} p_{ij}[c_{ij}-s]^+\right]\\
&\ge \min_{t\in\mathbb{R}^+}\left[t + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_1}} p_{ij}[c_{ij}-t]^+\right] + \frac{1}{1-\alpha}\min_{r\in\mathbb{R}^+}\left[\sum_{(i,j)\in\mathcal{A}^{l_2}\setminus\mathcal{A}^{l_1}} p_{ij}[c_{ij}-r]^+\right]\\
&= \min_{t\in\mathbb{R}^+}\left[t + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_1}} p_{ij}[c_{ij}-t]^+\right]\\
&= \mathsf{CVaR}_\alpha^{l_1}
\end{aligned}$$

where we set r an arbitrarily large number to make the corresponding minimum value zero.

(HA3) Let us introduce t^* such that

$$\begin{aligned}
\mathsf{CVaR}_\alpha^{l}(p,c) &= \min_{t\in\mathbb{R}^+}\left[t + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l}} p_{ij}[c_{ij}-t]^+\right]\\
&= t^* + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l}} p_{ij}[c_{ij}-t^*]^+
\end{aligned}$$

and s^* such that

$$\begin{aligned}\mathsf{CVaR}^l_\alpha(p+m,c+n) &= \min_{s\in\mathbb{R}^+}\left[s+\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^l}(p_{ij}+m_{ij})[c_{ij}+n_{ij}-s]^+\right]\\ &= s^*+\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^l}(p_{ij}+m_{ij})[c_{ij}+n_{ij}-s^*]^+\end{aligned}$$

Then we obtain the following relations:

$$\begin{aligned}\mathsf{CVaR}^l_\alpha(p+m,c+n) &= s^*+\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^l}(p_{ij}+m_{ij})[c_{ij}+n_{ij}-s^*]^+\\ &\geq s^*+\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^l}p_{ij}[c_{ij}-s^*]^+\\ &\geq t^*+\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^l}p_{ij}[c_{ij}-t^*]^+\\ &= \mathsf{CVaR}^l_\alpha(p,c)\end{aligned}$$

□

Proof of Proposition 4: The coherence of WCVaR is assured by Proposition 3. We prove the rest.

(HA1) When path l_2 contains path l_1, we have

$$\begin{aligned}\mathsf{WCVaR}^{l_2}_\alpha(\mathcal{U}_p,\mathcal{V}_c) &= \min_{r\in\mathbb{R}^+}\max_{\tilde{p}\in\mathcal{U}_p,\tilde{c}\in\mathcal{V}_c}\left\{r+\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_2}}\tilde{p}_{ij}[\tilde{c}_{ij}-r]^+\right\}\\ &= \min_{r\in\mathbb{R}^+}\max_{\tilde{p}\in\mathcal{U}_p,\tilde{c}\in\mathcal{V}_c}\left\{r+\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_1}}\tilde{p}_{ij}[\tilde{c}_{ij}-r]^+ +\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_2}\setminus\mathcal{A}^{l_1}}\tilde{p}_{ij}[\tilde{c}_{ij}-r]^+\right\}\\ &\leq \min_{s\in\mathbb{R}^+}\max_{\tilde{p}\in\mathcal{U}_p,\tilde{c}\in\mathcal{V}_c}\left\{s+\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_1}}\tilde{p}_{ij}[\tilde{c}_{ij}-s]^+\right\}\\ &\qquad +\frac{1}{1-\alpha}\min_{t\in\mathbb{R}^+}\max_{\tilde{p}\in\mathcal{U}_p,\tilde{c}\in\mathcal{V}_c}\left\{\sum_{(i,j)\in\mathcal{A}^{l_2}\setminus\mathcal{A}^{l_1}}\tilde{p}_{ij}[\tilde{c}_{ij}-t]^+\right\}\\ &= \min_{s\in\mathbb{R}^+}\max_{\tilde{p}\in\mathcal{U}_p,\tilde{c}\in\mathcal{V}_c}\left\{s+\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}^{l_1}}\tilde{p}_{ij}[\tilde{c}_{ij}-s]^+\right\}\\ &= \mathsf{WCVaR}^{l_1}_\alpha(\mathcal{U}_p,\mathcal{V}_c)\end{aligned}$$

where we set t an arbitrarily large number.

(HA3) For both cases of **(HA3')** and **(HA3")**, we can easily provide proofs that are similar to the CVaR measure case in Proposition 2.

□

Proof of Lemma 1: It is well known that a convex function retains its maximum at an extreme point if the feasible set is convex and compact (Pardalos and Rosen 1986). The objective function in (18) is a convex function of v and the set $\mathcal{V}$ is a polytope. Therefore there exists a solution v at an extreme point of $\mathcal{V}$, which is binary from the assumption that Γ_v is a positive integer, for any given r, x, and u. □

Proof of Proposition 5: Let us consider any binary x', u', and v', and the following problem:

$$\min_{r\in\mathbb{R}^+}\left(r+\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}}(p_{ij}+q_{ij}u'_{ij})[(c_{ij}+d_{ij}v'_{ij})-r]^+x'_{ij}\right) \tag{25}$$

Let $\mathcal{R}$ be an ordered set of $\{0\}\cup\{c_{ij}:(i,j)\in\mathcal{A}\}\cup\{c_{ij}+d_{ij}:(i,j)\in\mathcal{A}\}$ and r_k denote k-th smallest element of $\mathcal{R}$, where $\hat{r}_0=0$. Then the problem (25) is equivalent to the following problem:

$$\min_{k\in\{0,1,\ldots,|\mathcal{A}|-1\}}\ \min_{r_k\in[r_k,\hat{r}_{k+1}]}\left(r_k+\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}_1(r_k)}(p_{ij}+q_{ij}u'_{ij})[(c_{ij}+d_{ij}v'_{ij})-r_k]x'_{ij}\right) \tag{26}$$

where $\mathcal{A}_1(r_k)=\{(i,j):c_{ij}+d_{ij}v'_{ij}>r_k\}$. Note that we eliminated the '+' operator in (26). For each k, the inner problem of r in (26) is a linear optimization problem with a simple closed interval constraint. Therefore, r_k^* is either r_k or $\hat{r}_{k+1}$. Since x^*, u^*, and v^* to the WCVaR minimization problem (17) are all binary and the choice of x',u', and v' are arbitrary, we obtain the theorem. □

Proof of Proposition 6: For any path l, when $\alpha\to 0$, we have:

$$\begin{aligned}\mathsf{WCVaR}^l_\alpha &= \min_{r\in\mathbb{R}^+}\left(r+\frac{1}{1-\alpha}\max_{\tilde{p}\in\mathcal{U}_p,\tilde{c}\in\mathcal{V}_c}\sum_{(i,j)\in\mathcal{A}}\tilde{p}_{ij}[\tilde{c}_{ij}-r]^+\right)\\ &= \min_{r\in\mathbb{R}^+}\max_{\tilde{p}\in\mathcal{U}_p,\tilde{c}\in\mathcal{V}_c}\sum_{(i,j)\in\mathcal{A}^l}\left(\frac{1}{|\mathcal{A}^l|}r+\tilde{p}_{ij}[\tilde{c}_{ij}-r]^+\right)\end{aligned}$$

Let us consider the two cases: when $r>0$ and $r=0$.

- Case 1: $r>0$. In this case we have:

$$\begin{aligned}&\max_{\tilde{p}\in\mathcal{U}_p,\tilde{c}\in\mathcal{V}_c}\sum_{(i,j)\in\mathcal{A}^l}\left(\frac{1}{|\mathcal{A}^l|}r+\tilde{p}_{ij}[\tilde{c}_{ij}-r]^+\right)\\ &\geq \max_{\tilde{p}\in\mathcal{U}_p,\tilde{c}\in\mathcal{V}_c}\sum_{(i,j)\in\mathcal{A}^l}\left(\frac{1}{|\mathcal{A}^l|}r+\tilde{p}_{ij}(\tilde{c}_{ij}-r)\right)\\ &= \max_{\tilde{p}\in\mathcal{U}_p,\tilde{c}\in\mathcal{V}_c}\left(\sum_{(i,j)\in\mathcal{A}^l}\tilde{p}_{ij}\tilde{c}_{ij}+r\left(1-\sum_{(i,j)\in\mathcal{A}^l}\tilde{p}_{ij}\right)\right)\\ &> \max_{\tilde{p}\in\mathcal{U}_p,\tilde{c}\in\mathcal{V}_c}\sum_{(i,j)\in\mathcal{A}^l}\tilde{p}_{ij}\tilde{c}_{ij}\end{aligned}$$

because $1-\sum_{(i,j)\in\mathcal{A}^l}\tilde{p}_{ij}>0$.

- Case 2: $r=0$. In this case we have:

$$\begin{aligned}&\max_{\tilde{p}\in\mathcal{U}_p,\tilde{c}\in\mathcal{V}_c}\sum_{(i,j)\in\mathcal{A}^l}\left(\frac{1}{|\mathcal{A}^l|}r+\tilde{p}_{ij}[\tilde{c}_{ij}-r]^+\right)\\ &= \max_{\tilde{p}\in\mathcal{U}_p,\tilde{c}\in\mathcal{V}_c}\sum_{(i,j)\in\mathcal{A}^l}\tilde{p}_{ij}\tilde{c}_{ij}\end{aligned}$$

Comparing the two cases, we can conclude that the optimal solution when $\alpha\to 0$ can be obtained at $r=0$, and the WCVaR model is equivalent to the worst-case TR model. □

Proof of Proposition 7: We consider

$$\mathsf{WCVaR}^l_\alpha = \min_{r\in\mathbb{R}^+}\left(r + \frac{1}{1-\alpha}\max_{\tilde{p}\in\mathcal{U}_p,\tilde{c}\in\mathcal{V}_c}\sum_{(i,j)\in\mathcal{A}}\tilde{p}_{ij}[\tilde{c}_{ij}-r]^+\right)$$

which can be seen as a weighted sum of r and $\max_{\tilde{p}\in\mathcal{U}_p,\tilde{c}\in\mathcal{V}_c}\sum_{(i,j)\in\mathcal{A}}\tilde{p}_{ij}[\tilde{c}_{ij}-r]^+$. When $\alpha\to 1$, the weight for the second term becomes very large and dominant. Therefore, to minimize the weighted sum, we can determine the solution

$$r^* = \max_{\tilde{c}\in\mathcal{V}_c}\max_{(i,j)\in\mathcal{A}^l}\tilde{c}_{ij} \tag{27}$$

so that the second term vanishes. For any r greater than (27), the second term vanishes, but the choice of (27) also minimizes the first term. Hence,

$$\mathsf{WCVaR}^l_\alpha = \max_{\tilde{c}\in\mathcal{V}_c}\max_{(i,j)\in\mathcal{A}^l}\tilde{c}_{ij} = \max_{(i,j)\in\mathcal{A}^l}(c_{ij}+d_{ij})$$

for any $l\in\mathcal{P}$ when $\alpha\to 1$. □

References

Akgun, V., A. Parekh, R. Batta, C. Rump. 2007. Routing of a hazmat truck in the presence of weather systems. *Computers & Operations Research* **34**(5) 1351–1373.

Androutsopoulos, K. N., K. G. Zografos. 2012. A bi-objective time-dependent vehicle routing and scheduling problem for hazardous materials distribution. *EURO Journal on Transportation and Logistics* **1**(1-2) 157–183.

Artzner, P., F. Delbaen, J. Eber, D. Heath. 1999. Coherent measures of risk. *Mathematical Finance* **9**(3) 203–228.

Batta, R., S. S. Chiu. 1988. Optimal obnoxious paths on a network: Transportation of hazardous materials. *Operations Research* **36**(1) 84–92.

Bell, M. G. H. 2007. Mixed routing strategies for hazardous materials: Decision-making under complete uncertainty. *International Journal of Sustainable Transportation* **1**(2) 133–142.

Bertsimas, D., M. Sim. 2003. Robust discrete optimization and network flows. *Mathematical Programming* **98**(1) 49–71.

Carotenuto, P., S. Giordani, S. Ricciardelli. 2007. Finding minimum and equitable risk routes for hazmat shipments. *Computers & Operations Research* **34**(5) 1304–1327.

Čerbáková, J. 2006. Worst-case VaR and CVaR. *Operations Research Proceedings 2005*. Springer, 817–822.

Chaerani, D., C. Roos, A. Aman. 2005. The robust shortest path problem by means of robust linear optimization. H. Fleuren, D. Hertog, P. Kort, eds., *Operations Research Proceedings 2004*, *Operations Research Proceedings*, vol. 2004, chap. 42. Springer Berlin Heidelberg, Berlin/Heidelberg, 335–342–342.

Chang, T., L. Nozick, M. Turnquist. 2005. Multiobjective path finding in stochastic dynamic networks, with application to routing hazardous materials shipments. *Transportation Science* **39**(3) 383.

Chen, A., Z. Zhou. 2010. The α -reliable mean-excess traffic equilibrium model with stochastic travel times. *Transportation Research Part B* **44** 493–513.

Duffie, D., J. Pan. 1997. An overview of value at risk. *The Journal of Derivatives* **4**(3) 7–49.

Einhorn, D. 2008. Private profits and socialized risk. *Global Association of Risk Professionals Risk Review* **June/July**.

El Ghaoui, L., M. Oks, F. Oustry. 2003. Worst-case value-at-risk and robust portfolio optimization: A conic programming approach. *Operations Research* **51**(4) 543–556.

Emergency Response Guidebook. 2012. *2012 Emergency Response Guidebook : A guidebook for first responders during the initial phase of a dangerous goods/hazardous materials transportation incident*. U.S. Dept. of Transportation, Pipeline and Hazardous Materials Safety Administration, Washington, D.C. URL `http://phmsa.dot.gov/pv_obj_cache/pv_obj_id_7410989F4294AE44A2EBF6A80ADB640BCA8E4200/filename/ERG2012.pdf`.

Erkut, E., S. Tjandra, V. Verter. 2007. Hazardous materials transportation. C. Barnhart, G. Laporte, eds., *Transportation*, *Handbooks in Operations Research & Management Scienc*, vol. 14, chap. 9. North Holland, 539–611.

Erkut, E., V. Verter. 1998. Modeling of transport risk for hazardous materials. *Operations Research* 625–642.

Erkut, E., A. Ingolfsson. 2005. Transport risk models for hazardous materials: revisited. *Operations Research Letters* **33**(1) 81–89.

Federal Motor Carrier Safety Administration. 2001. *Comparative Risks of Hazardous Materials and Non-Hazardous Materials Truck Shipment Accidents/Incidents*. `http://www.fmcsa.dot.gov/documents/hazmatriskfinalreport.pdf`.

Federal Motor Carrier Safety Administration. 2007. *Final Report: Designation of Highway Routes for Hazardous Materials Shipments: Literature Review*. U.S. Department of Transportation. Prepared by Battelle.

Floudas, C. A., V. Visweswaran. 1994. Quadratic optimization. R. Horst, P. M. Pardalos, eds., *Handbook of Global Optimization*. Kluwer Academic Publishers, 217–270.

Jin, H., R. Batta. 1997. Objectives derived from viewing hazmat shipments as a sequence of independent bernoulli trials. *Transportation Science* **31**(3) 252–261.

Kang, Y., R. Batta, C. Kwon. 2014a. Value at risk model for hazardous material transportation. *Annals of Operations Research* **222**(1) 361–387.

Kang, Y., R. Batta, C. Kwon. 2014b. Generalized route planning model for hazardous material transportation with VaR and equity considerations. *Computers & Operations Research* **43** 237–247.

Kouvelis, P., G. Yu. 1996. *Robust Discrete Optimization and Its Applications (Nonconvex Optimization and Its Applications)*. 1st ed. Springer.

Kwon, C., T. Lee, P. Berglund. 2013. Robust shortest path problems with two uncertain multiplicative cost coefficients. *Naval Research Logistics* **60**(5) 375–394.

Linsmeier, T., N. Pearson. 2000. Value at risk. *Financial Analysts Journal* **56**(2) 47–67.

Miller-Hooks, E., H. Mahmassani. 1998. Optimal routing of hazardous materials in stochastic, time-varying transportation networks. *Transportation Research Record: Journal of the Transportation Research Board* **1645**(-1) 143–151.

Nocera, J. 2009. Risk mismanagement. *The New York Times Magazine* **January 4, 2009**.

Pardalos, P., J. Rosen. 1986. Methods for global concave minimization: A bibliographic survey. *SIAM Review* **28**(3) 367–379.

Pflug, G. 2000. *Probabilistic Constrained Optimization: Methodology and Applications*. Kluwer Academic Publishers.

Pipeline and Hazardous Materials Safety Administration. 2013. URL `http://phmsa.dot.gov/hazmat/glossary`.

Rockafellar, R. T., S. Uryasev. 2000. Optimization of conditional value-at-risk. *Journal of Risk* **2**(3) 21–42.

Rockafellar, R., S. Uryasev. 2002. Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance* **26**(7) 1443 – 1471.

Sarykalin, S., G. Serraino, S. Uryasev. 2008. Value-at-risk vs. conditional value-at-risk in risk management and optimization. *Tutorials in Optimization Research* 270–294.

Shaver, D., M. Kaiser. 1998. *Criteria for highway routing of hazardous materials*. Transportation Research Board, National Cooperative Highway Research Program.

Sherali, H., L. Brizendine, T. Glickman, S. Subramanian. 1997. Low probability–high consequence considerations in routing hazardous material shipments. *Transportation Science* **31**(3) 237–251.

Toumazis, I., C. Kwon. 2013. Routing hazardous materials on time-dependent networks using conditional value-at-risk. *Transportation Research Part C: Emerging Technologies* **37** 73 – 92.

Toumazis, I., C. Kwon, R. Batta. 2013. Value-at-risk and conditional value-at-risk minimization for hazardous materials routing. R. Batta, C. Kwon, eds., *Handbook of OR/MS Models in Hazardous Materials Transportation*, *International Series in Operations Research & Management Science*, vol. 193. Springer New York, 127–154.

U.S. Census Bureau. 2007. *United States: 2007, Hazardous Materials Transportation - Commodity Flow Survey*. Http://www.bts.gov/programs/commodity_flow_survey.

US Census Bureau. 2010. URL `http://www.census.gov/2010census`.

Waller, T., A. Ziliaskopoulos. 2006. A chance-constrained based stochastic dynamic traffic assignment model: Analysis, formulation and solution algorithms. *Transportation Research Part C: Emerging Technologies* **14**(6) 418–427.

Wu, X., Y. M. Nie. 2011. Modeling heterogeneous risk-taking behavior in route choice: A stochastic dominance approach. *Transportation Research Part A: Policy and Practice* **45**(9) 896 – 915.

Zhu, S., M. Fukushima. 2009. Worst-case conditional value-at-risk with application to robust portfolio management. *Operations Research* **57**(5) 1155–1168.

Zografos, K., K. Androutsopoulos. 2008. A decision support system for integrated hazardous materials routing and emergency response decisions. *Transportation Research Part C: Emerging Technologies* **16**(6) 684–703.